

THE PROPAGATION OF CHAOS FOR A RAREFIED GAS OF HARD SPHERES IN THE WHOLE SPACE

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ABSTRACT. We discuss old and new results on the mathematical justification of Boltzmann's equation. The classical result along these lines is a theorem which was proven by Lanford in the 1970s. This paper is naturally divided into three parts.

I. Classical. We give new proofs of both the uniform bounds required for Lanford's theorem, as well as the related bounds due to Illner & Pulvirenti for a perturbation of vacuum. The proofs use a duality argument and differential inequalities, instead of a fixed point iteration.

II. Strong chaos. We introduce a new notion of propagation of chaos. Our notion of chaos provides for uniform error estimates on a very precise set of points; this set is closely related to the notion of strong (one-sided) chaos and the emergence of irreversibility.

III. Supplementary material. We announce and provide a sketch of proof (in Appendix A) of propagation of *partial* factorization at some phase-points where complete factorization is impossible. In Appendix B we give a new discussion of some classical observations due to Illner.

1. INTRODUCTION

We are interested in the system of N identical elastic hard spheres of diameter $\varepsilon > 0$, which move through d -dimensional Euclidean space according to the laws of Newtonian mechanics. This is an important model in mathematical physics because the rules are relatively simple and yet they capture in a realistic way the macroscopic behavior of many physical systems. Usually the number of particles is quite large, say $N = 10^{23}$, so it seems hopeless to follow the microscopic dynamics directly. An alternative strategy, pioneered by Maxwell and Boltzmann, is to assign probabilities to the possible microscopic configurations of the system and study the evolution of these probabilities subject to mechanistic laws (e.g., conservation of mass, momentum and energy). Given a suitable choice of spatial and temporal scales, the equation one formally arrives at through this line of reasoning is known as Boltzmann's equation.

Half a century after Boltzmann's work, H. Grad used precise physical reasoning in an attempt to give Boltzmann's equation a firm physical footing. He devised a special scaling limit, known today as the *Boltzmann-Grad limit*, in which the microscopic dynamics heuristically reduce to the Boltzmann equation under an ill-defined "molecular chaos" assumption. [18] However, this did not resolve the question of deriving Boltzmann's equation because

there was no *mathematical* argument linking the microscopic Liouville equation to the Boltzmann equation. O.E. Lanford provided such a link in the 1970's, by describing the reduced dynamics arising from low-order correlations, and showing that the high-order correlations have negligible influence on the behavior of the gas, at least for a short time. [26] More recently, a careful *quantitative* analysis of Lanford's theorem has been provided by I. Gallagher, L. Saint-Raymond and B. Texier. [15]

We remark on several related developments. The major limitation in Lanford's theorem is the short time of validity, which so far has not been lifted except in very restrictive perturbative regimes. R. Illner and M. Pulvirenti were able to overcome the time restriction and prove global convergence for a highly rarefied gas in vacuum, using inequalities related to the dispersive nature of the system. [20–22] H. van Beijeren, O. E. Lanford, J. Lebowitz and H. Spohn studied the evolution of N particles at equilibrium in a box; they considered an initial perturbation which alters just *one* particle's state, while leaving all the other particles unperturbed at the initial time (though all particles interact under the dynamics). They found that the distribution for the “tagged” particle evolves according to the linear Boltzmann equation, while the remaining particles remain at equilibrium; this was eventually proven by J. Lebowitz and H. Spohn for arbitrary time intervals. [27, 35] More recently, T. Bodineau, I. Gallagher and L. Saint-Raymond were able to prove quantitative estimates and thereby pass to a diffusive scaling regime, showing that the mutual interaction of a tagged particle with a gas at equilibrium would converge to a Brownian motion for the tagged particle. [5] The same authors have also analyzed a more symmetric N -particle distribution in order to derive the *linearized* Boltzmann equation. [4]

There are several other important results which are not directly related to Lanford's theorem but are nevertheless foundational in kinetic theory.

- *Stochastic models.* All models we have mentioned so far have been fully deterministic; this means that randomness is allowed in the choice of initial data, but the *evolution* for each initial state is fully determined. However, there is an important class of models in kinetic theory where the dynamics itself introduces randomness. We specifically mention the Kac model; in this model, the position coordinates are treated as hidden variables, and in particular the impact parameter for each collision is a random variable with some specified law. When the number of particles tends to infinity, the evolution is seen to converge to the (nonlinear) space-homogeneous Boltzmann equation with the appropriate collision kernel. These models were first analyzed in a couple of influential papers by M. Kac and H. McKean. [24, 28] There have been many papers dealing with similar models in the intervening years, and a very complete treatment has been given by S. Mischler and C. Mouhot. [30]

- *Lorentz gases.* We refer to a class of models first studied by G. Gallavotti. [16] In these Lorentz gas-type models, the dynamics is indeed deterministic, but they differ from the case of Lanford in that all the particles but one are considered *stationary obstacles*, distributed like Poisson scatterers. The dynamics is much simpler in this case because the background particles never move out of place; in the Boltzmann-Grad limit one recovers the linear Boltzmann equation for the evolution of the tagged particle. Note that it is not possible to enforce momentum conservation in a Lorentz gas, so these models are only physically realistic if the tagged particle is much lighter than the background particles.
- *Vlasov-type mean field limit.* Physical limits in which each particle feels the influence of the entire gas are generally called mean-field limits; these models can be fully deterministic, or they can possess some stochasticity. Mean field limits tend to have a relatively pleasant mathematical structure because a typical particle's trajectory is governed by the *average* of the other particles' trajectories; this property is very helpful in controlling the correlations generated by the dynamics. Whereas the Boltzmann-Grad scaling leads to Boltzmann's kinetic equation, the Vlasov-type mean-field models lead to Vlasov-type equations in the limit $N \rightarrow \infty$. The study of Vlasov-type mean field limits is a vast field in its own right and we provide only a small sampling of the relevant literature. [14, 23, 29]

Besides the results discussed above, there have also been a few major results for *space-inhomogeneous* stochastic models in kinetic or hydrodynamic scalings. [31, 34] Henceforth in this work we will not be concerned with stochastic models, Lorentz gases, or mean field limits.

The goals of the present work are twofold. First, we shall present a new proof of the uniform bounds which are central to Lanford's theorem. We use differential inequalities and a duality argument, instead of a fixed point argument, to control the growth of correlations in the BBGKY hierarchy. We will apply this method to prove both the short-time result of Lanford, as well as the global near-vacuum result of Illner & Pulvirenti. [21, 22, 26] Our second goal is to thoroughly address the issue of uniform convergence of the marginals in the limit $N \rightarrow \infty$. The motivation is the notion of *strong (one-sided) chaos* and the appearance of irreversibility from an underlying reversible dynamics.¹ The issue of irreversibility is tied to convergence properties along very singular sets in phase space; for this reason, uniform

¹As has been pointed out in [6], chaoticity is not a necessary condition for irreversible behavior, and weak versions of chaos do not imply irreversible behavior. However, *strong* chaos guarantees irreversibility and may be propagated by the dynamics. Our theorem on strong chaos naturally generalizes to include non-chaotic data in the spirit of the Hewitt-Savage theorem. [19]

convergence (on a sufficiently large set) becomes a central question in the discussion of irreversibility.²

Uniform convergence has been addressed by a number of authors going back to the 1970s. (See [6, 25, 32], and Appendix A of [35].) Unlike previous results (except [6], which represents independent concurrent work), our strong chaos result implies uniform error estimates up to the *boundary* of the reduced phase space, which is significant because the physical interaction is confined to the boundary.³ Our error estimates are quantitative, as in [15, 32], though for simplicity of presentation we will state our main theorems without explicit rates. A novel aspect of our analysis is that, given suitably prepared initial data, we can propagate *partial* factorization even at phase points where complete factorization necessarily fails (i.e. “post-collisional” configurations with $t > 0$). As an application of our result, one obtains the existence of positive measure sets, parameterized by ε in a natural way, with measure tending to zero as $\varepsilon \rightarrow 0$, upon which $f_N^{(3)} \approx f_N^{(2)} \otimes f_N^{(1)}$ but further factorization is impossible. The proof of partial factorization draws significant inspiration from [4, 33], though our methods are somewhat different. Partial factorization is easily generalized to include *non-chaotic* initial data, in the spirit of the Hewitt-Savage theorem. [19] We emphasize that non-chaotic initial states have been discussed in the context of irreversibility; see, e.g., [6].

Organization of the paper. In Section 2, we describe the ideas behind our proof, and we present our main convergence result. Section 3 gives the precise physical setting for our problem, along with a crucial comparison principle. Section 4 briefly introduces the BBGKY and dual BBGKY hierarchies. Section 5 & 6 give proofs of *a priori* bounds on the BBGKY hierarchy by a duality argument; bounds are proven both locally in time for large data, and globally in time for data sufficiently close to vacuum. (These *a priori* estimates are not new, but we use a different approach for the proofs.) Sections 7, 8, 9, 10 & 11 introduce a number of important technical tools and results; our main technical contribution is the stability result in Section 8. The detailed convergence proof (part (i) of Theorem 2.1) is given in Section 12. An abbreviated proof of part (ii) of Theorem 2.1 is presented in Appendix A. An interesting proposition, which can be traced back to Illner [20], is proven in Appendix B.

²We would like to thank L. Saint-Raymond and H. Spohn (*private communications*) for insightful discussions and comments regarding the connection to irreversibility.

³Our proof formally extends even to points *on* the boundary, though this requires extra care because the trace along the boundary need not exist. Note that since the marginals solve a transport equation with locally bounded right-hand side, the trace is at least well-defined (a.e. $\omega \in \mathbb{S}^{d-1}$) for *almost every* time and any N . It remains to show that this trace is given explicitly in terms of a certain perturbative series; we omit the details.

2. STATEMENT OF MAIN RESULTS

2.1. Uniform bounds via duality. We begin by briefly describing the role that duality plays in our proof. Throughout this work we will rely on the BBGKY hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon), which is a sequence of equations describing the evolution of marginals $f_N^{(s)}(t)$ under the hard sphere flow. One of the key steps in the proof of Lanford's theorem is to bound a weighted L^∞ norm of the sequence of marginals, uniformly in N , in terms of the initial data. Lanford proves the uniform bound by re-writing the BBGKY hierarchy using Duhamel's formula and then setting up a fixed point argument.⁴ [15, 26] We have approached the uniform bounds from a somewhat different point of view. Our starting point is the *dual BBGKY hierarchy*, which is (formally) the semigroup whose generator is the (formal) adjoint of the semigroup generator for the BBGKY hierarchy. We refer to [10, 17] for background and results concerning the dual BBGKY hierarchy.

Physically, the dual BBGKY hierarchy describes the evolution of *observables*. We are able to bound the growth of observables in a weighted \mathcal{L}^1 space; then, the classical L^∞ bounds on the marginals follow by duality. Using the same strategy, with slight revisions, we are able to prove uniform bounds *globally in time* in the physical regime considered by Illner & Pulvirenti. [21, 22] We emphasize that all of our results concerning uniform bounds are classical; the only novelty lies in the method of proof. Note that certain very special observables, such as the kinetic energy, exhibit cancellation properties (e.g. conservation). However, our proof concerns *generic* observables; in particular, there seems to be no simple way to account for cancellations. Hence we cannot report any improvements beyond the perturbative regime (small time or large mean free path).

Remark. One can ask whether it is possible to treat the Boltzmann hierarchy using duality, in a manner similar to the BBGKY hierarchy. The answer is “yes, but...” The problem is that, whereas the dual BBGKY hierarchy propagates \mathcal{L}^1 regularity, solutions of the dual Boltzmann hierarchy are *measures* even if the data is smooth.⁵ Unfortunately, the dual Boltzmann hierarchy isn't well-defined for measure data due to the possibility of simultaneous collisions of three or more particles. Most likely it is possible to work with the dual Boltzmann hierarchy by restricting one's attention to measures that assign zero weight to manifolds of sufficiently high codimension. However, we prefer not to confront these technical issues; instead, we prove uniform bounds for the Boltzmann hierarchy using the standard fixed-point argument.

⁴In fact Lanford wrote out a series expansion for which he proved L^∞ bounds uniformly in N ; this is effectively equivalent to the fixed point argument, and he had to prove the same collision operator bounds as in [15].

⁵This is related to the fact that the Boltzmann hierarchy is not well-posed on (weighted) L^∞ but is well-posed for *continuous* data.

Remark. An interesting problem would be to prove propagation of chaos by comparing a suitable pair of *dual* semigroup generators. The main advantage of this approach over Lanford’s is that the analysis is based on differential inequalities (as is our proof of uniform bounds), so there is no need for the analysis of complicated “pseudo-trajectories” going back to the initial data. In this setting, the concept of strong (one-sided) chaos is replaced by a notion of one-sided regularity for observables; here, regularity means that certain pathological sets have small weight. Though we do not complete this program in the current work, the proof of propagation of chaos via duality is a topic of ongoing research.

2.2. Strong convergence. We now turn to the content of Theorem 2.1 (especially part (i)), which is our main new result. Essentially the result states that if *a priori* bounds are known then chaoticity is propagated forwards in time; the novelty of the result lies in the strength of the notion of convergence we employ at positive times. The direction of time is built into our notion of chaoticity, so the theorem *cannot* be applied to prove propagation of chaos backwards in time. Our convergence result is a type of *strong chaos* result; this means that we can take the evolved state at a time $t > 0$ and use this state as *initial data* in order to iterate the convergence to an even later time. The iteration can be continued as long as uniform bounds are known. We emphasize that strong chaos results are known in both the classical and very recent literature [6, 35]; however, to our knowledge, the present convergence result is the only one which extends to *all* distance scales $\{|x_i - x_j| > \varepsilon\}$ as long as the backwards trajectories of all s particles are free (minus a small set in the *velocity* variables only, see part (i) of Theorem 2.1).⁶ Moreover, as we will see, our proof extends without too much difficulty to obtain a *pointwise* description of two-particle correlations (higher-order correlations and better error estimates are topics of ongoing research).⁷

Remark. The notion of chaos that Lanford originally proved (at positive times) states that the marginals $f_N^{(s)}(t)$ converge pointwise almost everywhere to tensor products. It can be shown (see [26]) that this notion of chaos (combined with certain uniform estimates) implies that for any box $\Delta \subset \mathbb{R}^d \times \mathbb{R}^d$, the occupation fraction

$$\frac{1}{N}n_\Delta(t) = \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{(x_i(t), v_i(t)) \in \Delta} \quad (1)$$

converges in probability to a constant depending only on t and Δ when $\varepsilon \rightarrow 0$. The physical interpretation of Lanford’s result is that fluctuations tend to zero as $\varepsilon \rightarrow 0$. We emphasize that it is *not* true that if

⁶The strong chaos result in [6] requires $|x_i - x_j| \gtrsim \varepsilon \log \frac{1}{\varepsilon}$.

⁷By contrast, the authors of [33] have provided a very precise but *averaged* (not pointwise) description of correlations.

the marginals converge pointwise almost everywhere (at $t = 0$, or even for all $t \in [0, T]$), then the evolution is governed by Boltzmann's equation. The classical counter-example uses the reversibility of Newton's laws combined with the irreversibility of Boltzmann's equation. [11] An even more striking counter-example has been constructed by T. Bodineau, I. Gallagher, L. Saint-Raymond and S. Simonella; these authors found an initial data such that the marginals converge pointwise almost everywhere to tensor products at $t = 0$ (indeed they obtained *uniform* convergence off explicit small sets), whereas the evolution is given by *free transport*. [6]

We will need to introduce several sets before we can state our main result; to this end, we will borrow notation from Section 3. We will view $\eta > 0$ as a small velocity cut-off, and $R > 0$ as a large velocity cutoff.

$$\mathcal{K}_s = \{Z_s = (X_s, V_s) \in \overline{\mathcal{D}_s} \mid \psi_s^{-\tau} Z_s = (X_s - V_s \tau, V_s) \ \forall \tau > 0\} \subset \mathbb{R}^{2ds} \quad (2)$$

$$\mathcal{U}_s^\eta = \left\{ Z_s = (X_s, V_s) \in \overline{\mathcal{D}_s} \mid \inf_{1 \leq i < j \leq s} |v_i - v_j| > \eta \right\} \subset \mathbb{R}^{2ds} \quad (3)$$

$$\mathcal{G}_s = \left\{ Z_s = (X_s, V_s) \in \overline{\mathcal{D}_s} \mid \begin{array}{l} \forall \tau > 0, \ \forall 3 \leq i \leq s, \\ \quad (\psi_s^{-\tau} Z_s)_i = (x_i - v_i \tau, v_i) \\ \text{and, } \forall \tau > 0, \ \forall 1 \leq i \leq 2, \ \forall 3 \leq j \leq s, \\ \quad |(x_i - x_j) - (v_i - v_j) \tau| > \varepsilon \end{array} \right\} \quad (4)$$

$$\mathcal{V}_s^\eta = \left\{ (Z_s, Z'_s) \in \overline{\mathcal{D}_s} \times \overline{\mathcal{D}_s} \mid \begin{array}{l} \inf_{1 \leq i \neq j \leq s} |v_i - v'_j| > \eta \\ \text{and} \\ \inf_{1 \leq i \leq s : v_i \neq v'_i} |v_i - v'_i| > \eta \end{array} \right\} \quad (5)$$

$$\hat{\mathcal{U}}_s^\eta = \left\{ Z_s = (X_s, V_s) \in \mathcal{U}_s^\eta \mid \forall \tau, \tau' > 0, (\psi_s^{-\tau} Z_s, \psi_s^{-\tau'} Z_s) \in \mathcal{V}_s^\eta \right\} \quad (6)$$

We write $F_N(t) = \left\{ f_N^{(s)}(t, Z_s) \right\}_{1 \leq s \leq N}$ where each $f_N^{(s)}(t, Z_s)$ is a function on $[0, \infty) \times \overline{\mathcal{D}_s}$ which is symmetric under interchange of particles. Observe that $Z_s \in \mathcal{K}_s$ precisely when the backwards trajectories of all s particles are free. On the other hand, $Z_s \in \mathcal{G}_s$ precisely when the backwards trajectories of the last $s - 2$ particles are free (including the cases where the first two particles collide *or* simply “pass through” each other without colliding). This distinction will lead us to define two different notions of chaos, each of which yields a strong chaos result.

Definition 2.1. The sequence of initial data $\{F_N(0) \mid N \in \mathbb{N}\}$ is *nonuniformly f_0 -chaotic* if, for some $\kappa \in (0, 1)$, we have for all $s \in \mathbb{N}$ and all $R > 0$ that

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(0, Z_s) - f_0^{\otimes s}(Z_s) \right) \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (7)$$

where $\eta(\varepsilon) = \varepsilon^\kappa$ and $N\varepsilon^{d-1} = \ell^{-1}$.

Definition 2.2. The sequence of initial data $\{F_N(0) \mid N \in \mathbb{N}\}$ is *2-nonuniformly f_0 -chaotic* if, for all $R > 0$,

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(1)}(0, x, v) - f_0(x, v) \right) \mathbf{1}_{\frac{1}{2}|v|^2 \leq R^2} \right\|_{L_{x,v}^\infty} = 0 \quad (8)$$

and, for some $\kappa \in (0, 1)$, we have for all $s \in \mathbb{N}$ with $s \geq 3$ and all $R > 0$ that

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(0, Z_s) - \left(f_N^{(2)}(0) \otimes f_0^{\otimes(s-2)} \right) (Z_s) \right) \mathbf{1}_{Z_s \in \mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (9)$$

where $\eta(\varepsilon) = \varepsilon^\kappa$ and $N\varepsilon^{d-1} = \ell^{-1}$.

Remark. Observe that the sets \mathcal{G}_s appearing in Definition 2.2 are not symmetric under particle interchange. Nevertheless, since we assume that the marginals $f_N^{(s)}$ are symmetric, the uniform error estimates hold on the image of the set $\mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}$ under any permutation of particle labels.

Remark. The key difference between Definition 2.1 and Definition 2.2 is that the set $\mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}$ is replaced by $\mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}$. Hence, the estimate (7) holds only at phase points which possibly involve collisions in the *future*, but not the *past*. On the other hand, the estimate (9) holds even at points where *at most two* of the particles have collisions in the *past*. Also note that the structure of the set $\hat{\mathcal{U}}_s^{\eta(\varepsilon)}$ is more complicated than that of $\mathcal{U}_s^{\eta(\varepsilon)}$ due to its dependence on the particle flow ψ_s^{-t} .

Remark. It is important to realize that complete factorization is allowed even at positive times along all of $\mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}$, but only *partial* factorization is possible at some points of $\mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}$ when $t > 0$; this is due to the fact that collisions generate correlations. In this sense, 2-nonuniform chaoticity captures (very crudely) the fine-scale *structure of correlations* at positive times. There has been some recent interest in precisely characterizing the size of correlations in the Boltzmann-Grad limit; we refer to [4, 33] for some results along these lines. Compared to these previous results, the main difference with our result is that we draw a connection between correlations and strong chaos.

Recall the Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f(t) = \ell^{-1} Q(f(t), f(t)) \quad (10)$$

$$Q(f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} [\omega \cdot (v_1 - v)]_+ (f(x, v^*) f(x, v_1^*) - f(x, v) f(x, v_1)) d\omega dv_1 \quad (11)$$

We are able to show:

Theorem 2.1. *Suppose that the Boltzmann equation (10) has a non-negative solution $f(t)$ for $t \in [0, T]$, with $\int f(t) dx dv = 1$, and further suppose that*

there exists $\beta_T > 0$ such that

$$\sup_{0 \leq t \leq T} \sup_{x, v \in \mathbb{R}^d} e^{\frac{1}{2}\beta_T |v|^2} f(t, x, v) < \infty \quad (12)$$

and $f(t) \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ for $t \in [0, T]$. Let $F_N(t)$ solve the hard sphere BBGKY hierarchy, under the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, and suppose that there is a $\tilde{\beta}_T > 0$, $\tilde{\mu}_T \in \mathbb{R}$ such that

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} e^{\tilde{\beta}_T E_s(Z_s)} e^{\tilde{\mu}_T s} \left| f_N^{(s)}(t, Z_s) \right| < \infty \quad (13)$$

Then the following holds:

- (i) If $\{F_N(0)\}_N$ is nonuniformly f_0 -chaotic, then for all $t \in [0, T]$, $\{F_N(t)\}_N$ is nonuniformly $f(t)$ -chaotic (with the same κ).
- (ii) If $\{F_N(0)\}_N$ is 2-nonuniformly f_0 -chaotic, then for all $t \in [0, T]$, $\{F_N(t)\}_N$ is 2-nonuniformly $f(t)$ -chaotic (with the same κ).

We will prove in full detail part (i) of Theorem 2.1. The proof of part (ii) is similar to the proof of part (i); the main differences are the use of an intermediate (Boltzmann-Enskog) hierarchy, as in [33], combined with a refined analysis of pseudo-trajectories. We supply the necessary ideas and some key technical estimates for part (ii) in Appendix A; the remaining details are omitted for the sake of brevity. We remark that a more general version of part (ii) (accounting for correlations of any finite number of particles) is the topic of ongoing research.

Remark. The time T in Theorem 2.1 is not necessarily the time in Lanford's original theorem. For instance, in the case of a sufficiently small perturbation of vacuum [21, 22], it is permissible to take $T = \infty$. More generally, if the *a priori* estimate (13) is known for a *specific (tensorized) solution* of the BBGKY hierarchy up to time T , then we can propagate (2-)nonuniform chaoticity up to time T . Note that T is necessarily smaller than the existence time for classical solutions of the Boltzmann equation.

So far we have drawn a connection between a particular notion of chaos and irreversibility. However, chaoticity is *not* a necessary condition for irreversible behavior. [6, 11] Let us suppose that $f_N^{(s)}(t)$ are the marginals of an underlying N -particle probability density $f_N(t)$ which is symmetric under particle interchange. Assume that as $N \rightarrow \infty$, the marginals $f_N^{(s)}(t)$ converge to functions $f_\infty^{(s)}(t)$ which satisfy the properties of *non-negativity*, *normalization* and *consistency* (respectively): (these are all true at finite N in any case)

$$f_\infty^{(s)}(t) \geq 0 \quad (14)$$

$$\int_{\mathbb{R}^{2ds}} f_\infty^{(s)}(t) dZ_s = 1 \quad (15)$$

$$f_\infty^{(s)}(t, Z_s) = \int_{\mathbb{R}^{2d}} f_\infty^{(s+1)}(t, Z_{s+1}) dz_{s+1} \quad (16)$$

If the functions $\{f_\infty^{(s)}(t)\}_{s \in \mathbb{N}}$ are non-negative, normalized, and consistent, and symmetric under particle interchange, then the Hewitt-Savage theorem [19] tells us that there exists a time-dependent probability measure $\pi_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^{2d}))^8$ such that

$$f_\infty^{(s)}(t) = \int_{\mathcal{P}(\mathbb{R}^{2d})} h^{\otimes s}(Z_s) d\pi_t(h) \quad (17)$$

Hence, in very great generality, we are free to assume that the limiting distribution is a convex combination of factorized distributions. If the convergence of $\{f_N^{(s)}(t)\}_{1 \leq s \leq N}$ to $\{f_\infty^{(s)}(t)\}_{s \in \mathbb{N}}$ is sufficiently strong, and we have sufficient control on solutions to Boltzmann's equation, then it is possible to explicitly characterize the measure π_t .

It is possible to show the following result through a slight refinement (which we skip) of the proof of Theorem 2.1:

Theorem 2.2. *Let $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^{2d}))$. Suppose that for π -a.e. h_0 there exists a non-negative solution $h(t)$ of Boltzmann's equation on $[0, T]$ with $h(0) = h_0$, and with $\int h(t) dx dv = 1$, and further suppose that there exist $C_T, \beta_T > 0$ (which are constants on a set of full π -measure) such that*

$$\sup_{0 \leq t \leq T} \sup_{x, v \in \mathbb{R}^d} e^{\frac{1}{2}\beta_T |v|^2} h(t, x, v) \leq C_T \quad (18)$$

$$\sup_{0 \leq t \leq T} \|h(t)\|_{W^{1, \infty}(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_T \quad (19)$$

Let $F_N(t) = \{f_N^{(s)}(t)\}_{1 \leq s \leq N}$ solve the hard sphere BBGKY hierarchy, under the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$. Assume that there is a $\tilde{\beta}_T > 0$, $\tilde{\mu}_T \in \mathbb{R}$ such that

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} e^{\tilde{\beta}_T E_s(Z_s)} e^{\tilde{\mu}_T s} \left| f_N^{(s)}(t, Z_s) \right| < \infty \quad (20)$$

Suppose that for some $\kappa \in (0, 1)$, we have for all $s \in \mathbb{N}$ and all $R > 0$ that

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(0) - \int_{\mathcal{P}(\mathbb{R}^{2d})} h_0^{\otimes s} d\pi(h_0) \right) \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (21)$$

where $\eta(\varepsilon) = \varepsilon^\kappa$. Then for all $t \in [0, T]$, all $s \in \mathbb{N}$, and all $R > 0$ we have:

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(t) - \int_{\mathcal{P}(\mathbb{R}^{2d})} h(t)^{\otimes s} d\pi(h_0) \right) \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (22)$$

⁸Here $\mathcal{P}(X)$ is the set of Borel probability measures on the Polish space X .

Theorem 2.2 is a generalization of the propagation of nonuniform chaoticity when there is some uncertainty in the initial data h_0 for Boltzmann's equation. It is similarly possible to generalize the propagation of 2-nonuniform chaoticity to the situation where h_0 is random. However, one must be quite careful when dealing with 2-nonuniform chaoticity because the representation formula

$$\int_{\mathcal{P}(\mathbb{R}^{2d})} h(t)^{\otimes s} d\pi(h_0) \quad (23)$$

fails in general (when $t > 0$) at phase points for which a collision has occurred in the *past*. We have, by refinements (which we again skip) of the proof of Theorem 2.1, the following result:

Theorem 2.3. *Under the assumptions of Theorem 2.2, let us further suppose that for $\pi - a.e.$ h_0 we have sequences $H_N(t; h_0) = \{h_N^{(s)}(t; h_0)\}_{1 \leq s \leq N}$ such that $H_N(t; h_0)$ solves the hard sphere BBGKY hierarchy for $\pi - a.e.$ h_0 fixed, and $\{H_N(t; h_0)\}_N$ is 2-nonuniformly $h(t)$ -chaotic (with κ fixed once and for all) for each $t \in [0, T]$. (The existence of such sequences $\{H_N(t; h_0)\}_N$ can be proven on a short time interval using Theorem 2.1 and Lanford's uniform bounds.) Assume as in the statement of Theorem 2.2 that*

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} e^{\tilde{\beta}_T E_s(Z_s)} e^{\tilde{\mu}_T s} \left| h_N^{(s)}(t, Z_s; h_0) \right| < C \quad (24)$$

where C is constant on a set of full π -measure. Assume that

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(0) - \int_{\mathcal{P}(\mathbb{R}^{2d})} h_N^{(s)}(0; h_0) d\pi(h_0) \right) \mathbf{1}_{Z_s \in \mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (25)$$

where $\eta(\varepsilon) = \varepsilon^\kappa$. Then for all $t \in [0, T]$ we have

$$\limsup_{N \rightarrow \infty} \left\| \left(f_N^{(s)}(t) - \int_{\mathcal{P}(\mathbb{R}^{2d})} h_N^{(s)}(t; h_0) d\pi(h_0) \right) \mathbf{1}_{Z_s \in \mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq R^2} \right\|_{L_{Z_s}^\infty} = 0 \quad (26)$$

Remark. Theorem 2.3 gives us a new perspective on Lanford's theorem because it tells us how correlations of two particles are propagated. We emphasize that that, on the set $(\mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}) \setminus (\mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)})$, it is *not* possible to determine (even approximately) the functions $h_N^{(s)}(t; h_0)$ merely from the knowledge of $h(t)$. Therefore, the sequences $H_N(t; h_0)$ carry important information about $F_N(t)$ which is not available merely from the examination of π . We cannot just pick “any old” $\{H_N(0; h_0)\}_N$ which is 2-nonuniformly h_0 -chaotic; it must be chosen to be compatible with the entire history of the sequence F_N . Such compatibility conditions can be satisfied for very

particular initial data, such as that constructed in Section 11 (or convex combinations thereof), and states which are achieved from these through forward time evolution of the BBGKY hierarchy (at least for a short time).

Remark. It is very likely possible to track correlations of $m - 1$ particles and thereby prove propagation of some notion of $(m - 1)$ -nonuniform chaoticity. It should even be possible to send m to infinity and prove propagation of ∞ -nonuniform chaoticity. In the case of ∞ -nonuniform chaoticity, we would identify (for each s) specific N -dependent sets where $f_N^{(s)}(t) \approx f_N^{(s-1)}(t) \otimes f_t$. Unfortunately, these sets are *extremely* complicated because they depend on *all possible histories* of the $s - 1$ particles which are allowed to interact (including those histories where some of those $s - 1$ particles may be “deleted” from the interaction at some intermediate time, see Figure 1 in Appendix A). The cardinality of the set of possible histories is finite and quantitatively bounded for a given point Z_s and integer m by [8]; however, just writing down the correct set of convergence for the marginals is a task in itself. Therefore we have only considered the case $m = 3$ in the present work; larger values of m are a topic of ongoing research.

3. NOTATION AND A COMPARISON PRINCIPLE

We will work in the spatial domain \mathbb{R}^d for some $d \geq 2$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ satisfy the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ for some fixed parameter $\ell > 0$ ⁹; we will henceforth suppress the implicit dependence on ε, ℓ in our notation, though they will be retained in formulas and estimates. If $1 \leq s \leq N$ then we define the phase space

$$\mathcal{D}_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{ds} \times \mathbb{R}^{ds} \mid |x_i - x_j| > \varepsilon \forall 1 \leq i < j \leq s \right\} \quad (27)$$

Suppose $Z_s \in \partial\mathcal{D}_s$, with $x_j = x_i + \varepsilon\omega$, $\omega \in \mathbb{S}^{d-1}$, $\omega \cdot (v_j - v_i) \neq 0$, $i < j$, and $|x_{j'} - x_{i'}| > \varepsilon$ whenever $i' < j'$ and $(i', j') \neq (i, j)$; then the image point $Z_s^* = (x_1, v_1, \dots, x_i, v_i^*, \dots, x_j, v_j^*, \dots, x_s, v_s)$ is defined by the following rule:

$$\begin{cases} v_i^* = v_i + \omega\omega \cdot (v_j - v_i) \\ v_j^* = v_j - \omega\omega \cdot (v_j - v_i) \end{cases} \quad (28)$$

Note that the map $Z_s \mapsto Z_s^*$ is a measurable involution of $\partial\mathcal{D}_s$; and, in the identity $Z_s^{**} = Z_s$ a.e. $Z_s \in \partial\mathcal{D}_s$, we use the same $\omega \in \mathbb{S}^{d-1}$ for each transformation.

Let us denote by $\psi_s^t Z_s$ the image of Z_s under the forward time evolution of s hard spheres at time t ; that is, if $Z_s = Z_s(0)$, and the function $Z_s(t) = (X_s(t), V_s(t))$ is piecewise differentiable¹⁰ and has left and right limits at all

⁹The parameter ℓ is of order the mean free path length, insofar as the mean free path is well-defined.

¹⁰classically differentiable on the complement of a closed set of isolated points

points $t \in \mathbb{R}$, and there holds

$$\begin{cases} \frac{d}{dt}Z_s(t) = (V_s(t), 0) & \text{if } Z_s(t) \notin \partial\mathcal{D}_s \\ Z_s(t^+) = (Z_s(t^-))^* & \text{if } Z_s(t) \in \partial\mathcal{D}_s \end{cases} \quad (29)$$

for all $t \in \mathbb{R}$ then we write $\psi_s^t Z_s = Z_s(t)$. This “definition,” unfortunately, does *not* define $\psi_s^t Z_s$ uniquely in general, since there is no way to continuously extend the map $Z_s \mapsto Z_s^*$ to all of $\partial\mathcal{D}_s$. Indeed, discontinuities will be observed whenever one particle simultaneously collides with at least two other particles. Nevertheless, up to deletion of a Lebesgue measure zero subset of initial phase points $Z_s \in \mathcal{D}_s$, we may assume that $\psi_s^t Z_s$ is defined for all $t \in \mathbb{R}$, that all collisions are non-grazing, and that all collisions are binary and linearly ordered in time (i.e. disjoint *pairs* of particles do not simultaneously collide). [1] One can then show that, for each $t \in \mathbb{R}$, ψ_s^t may be viewed as a measurable map $\mathcal{D}_s \rightarrow \mathcal{D}_s$ preserving the induced Lebesgue measure. On bounded time intervals, the map $(t, Z_s) \mapsto \psi_s^t Z_s$ is actually *jointly continuous* away from certain higher codimension submanifolds of the domain, provided that one chooses to identify $Z_s \in \partial\mathcal{D}_s$ with its image Z_s^* . However, we will not make such an identification; instead, we choose to enforce the convention that, for a.e. $Z_s \in \mathcal{D}_s$, there holds for all $t \in \mathbb{R}$ that $\psi_s^t Z_s = \psi_s^{t+} Z_s$. We will say that a point $Z_s \in \partial\mathcal{D}_s$ is a *pre-collisional configuration* if $Z_s = \psi_s^{t-} Z_s$; or, we will call it a *post-collisional configuration* if $Z_s = \psi_s^{t+} Z_s$. Note in particular that, according to our conventions, $Z_s \neq \psi_s^0 Z_s$ for a.e. pre-collisional $Z_s \in \partial\mathcal{D}_s$.

Suppose $f_N(0)$ is a probability measure supported on $\overline{\mathcal{D}_N}$ and absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^{2dN} ; by abuse of notation, we call the corresponding density $f_N(0, Z_N)$. We will denote by \mathcal{S}_N the symmetric group on N letters; if $Z_N \in \mathcal{D}_N$ then $\sigma \in \mathcal{S}_N$ acts on Z_N by permutation of particle indices: $\sigma(z_1, \dots, z_N) = (z_{\sigma(1)}, \dots, z_{\sigma(N)})$. We will always assume that $f_N(0)$ is *symmetric*, i.e. for any $\sigma \in \mathcal{S}_N$ there holds $f_N(0, \sigma Z_N) = f_N(0, Z_N)$. Then for $t \in \mathbb{R}$ we will define $f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N)$; equivalently, since ψ_N^t preserves Lebesgue measure on \mathbb{R}^{2dN} , we can say that $f_N(t)$ is the pushforward of $f_N(0)$ under ψ_N^t . We will denote $Z_{s:s+k} = (z_s, z_{s+1}, \dots, z_{s+k})$, $Z_s^{(i)} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_s)$, and similarly $Z_{s:s+k}^{(i)}$ in the case $s \leq i \leq s+k$. We extend $f_N(t)$ by zero so that it is defined on \mathbb{R}^{2dN} ; then the marginals $f_N^{(s)}(t, Z_s)$ are defined by $f_N^{(s)}(t, Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_N) dZ_{(s+1):N}$. Each $f_N^{(s)}(t)$ is a symmetric probability density supported on $\overline{\mathcal{D}_s}$; and, since $f_N^{(s)}(t)$ is the marginal of $f_N^{(s+1)}(t)$ for each $1 \leq s < N$, we say that the sequence $\left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N}$ is *consistent*. We also define the energy $E_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |v_i|^2$, and we will also let $I_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |x_i|^2$, and additionally $\mathcal{Y}_s(Z_s) = \sum_{i=1}^s x_i \cdot v_i$.

Remark. Sometimes we will want to consider sequences $\left\{f_N^{(s)}\right\}_{1 \leq s \leq N}$ which are *not* consistent, and not necessarily normalized nor even non-negative. We will only point out this distinction when it is important for the analysis. For the remainder of this section, we will assume that $\left\{f_N^{(s)}\right\}_{1 \leq s \leq N}$ is a consistent sequence of symmetric probability densities.

We now turn to a comparison principle; this result is due to Illner & Pulvirenti [20–22] and is specific to the whole space case.

Lemma 3.1. *For a.e. $Z_s = (X_s, V_s) \in \mathcal{D}_s$ and all $t \geq 0$,*

$$\mathcal{Y}_s(\psi_s^t Z_s) \geq 2tE_s(Z_s) + \mathcal{Y}_s(Z_s) \quad (30)$$

Proof. Fix $Z_s \in \mathcal{D}_s$ such that $\psi_s^t Z_s$ is defined for all $t \in \mathbb{R}$, with all collisions binary and non-grazing. Let $r(t) = \mathcal{Y}_s(\psi_s^t Z_s) - 2tE_s(\psi_s^t Z_s)$; then $r(0) = \mathcal{Y}_s(Z_s)$. Between collisions we have $\frac{d}{dt}r(t) = 0$, and r can only increase across collisions. We use the energy conservation identity $E_s(\psi_s^t Z_s) = E_s(Z_s)$ to conclude. \square

Lemma 3.2. *For a.e. $Z_s = (X_s, V_s) \in \mathcal{D}_s$ and all $t \in \mathbb{R}$,*

$$I_s(\psi_s^t Z_s) \geq I_s((X_s + V_s t, V_s)) \quad (31)$$

Proof. Due to time reversibility, it suffices to consider the case $t \geq 0$. Fix $Z_s \in \mathcal{D}_s$ such that $\psi_s^t Z_s$ is defined for all $t \in \mathbb{R}$, with all collisions binary and non-grazing. Let $b(t) = I_s(\psi_s^t Z_s) - I_s((X_s + V_s t, V_s))$; observe that $b(0) = 0$, and $b(t)$ is continuous and piecewise smooth. Between collisions we have

$$\frac{d}{dt}b(t) = \mathcal{Y}_s(\psi_s^t Z_s) - 2tE_s(Z_s) - \mathcal{Y}_s(Z_s) \geq 0 \quad (32)$$

where we have used Lemma 3.1. Therefore $b(t) \geq 0$ for all $t > 0$, and the result follows. \square

4. THE BBGKY AND DUAL BBGKY HIERARCHIES

The BBGKY hierarchy is a sequence of equations which describe the evolution of the marginals $f_N^{(s)}(t)$ of a solution $f_N(t)$ of Liouville's equation. The BBGKY hierarchy is one of the classical tools in the mathematical analysis of many-particle systems. Many derivations of the BBGKY hierarchy have been devised; we refer to [15], which will be the approach most convenient for us. We give a slightly generalized version of the weak form of the BBGKY hierarchy derived in [15], since it will enable us to easily read off the *dual BBGKY hierarchy*. The dual BBGKY hierarchy is the sequence of equations whose semigroup generator is the adjoint of that of the BBGKY hierarchy. We will be using the dual BBGKY hierarchy in order to derive uniform bounds in Sections 5 and 6. The main advantage of the dual BBGKY hierarchy is that the semigroup generator makes sense without strong regularity assumptions; this is useful because the BBGKY hierarchy does not propagate smoothness of the marginals.

Suppose we are given a sequence of functions $\{f_N^{(s)}(t, Z_s)\}_{1 \leq s \leq N}$, with $f_N^{(s)}$ defined on $[0, \infty) \times \overline{\mathcal{D}_s}$ and $(\partial_t + V_s \cdot \nabla_{X_s}) f_N^{(s)} \in L^1(\mathcal{O})$ for any bounded open set $\mathcal{O} \subset [0, \infty) \times \overline{\mathcal{D}_s}$. Further suppose the marginals satisfy permutation symmetry and the boundary condition $f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s)$ a.e. $(t, Z_s) \in [0, \infty) \times \partial\mathcal{D}_s$. Then we will say that the sequence $\{f_N^{(s)}(t, Z_s)\}_{1 \leq s \leq N}$ *solves the weak form of the BBGKY hierarchy* provided that for every test function $\varphi_s(t, Z_s) \in C_c^1([0, \infty) \times \overline{\mathcal{D}_s})$, satisfying permutation symmetry, there holds:

$$\begin{aligned}
& \int_0^\infty \int_{\mathcal{D}_s} [(\partial_t + V_s \cdot \nabla_{X_s}) \varphi_s(t, Z_s)] f_N^{(s)}(t, Z_s) dZ_s dt = \\
& = \int_{\mathcal{D}_s} \varphi_s(0, Z_s) f_N^{(s)}(0, Z_s) dZ_s \\
& - \varepsilon^{d-1} \sum_{1 \leq i < j \leq s} \int_0^\infty \int_{\mathbb{R}^{ds} \times \mathbb{R}^{d(s-1)} \times \mathbb{S}^{d-1}} \mathbf{1}_{Z_s \in \partial\mathcal{D}_s} \omega \cdot (v_j - v_i) \times \\
& \quad \times \left(\varphi_s f_N^{(s)} \right) (t, \dots, x_i, v_i, \dots, x_i + \varepsilon\omega, v_j, \dots) d\omega dX_s^{(j)} dV_s dt \\
& - (N-s)\varepsilon^{d-1} \sum_{1 \leq i \leq s} \int_0^\infty \int_{\mathbb{R}^{ds} \times \mathbb{R}^{ds} \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \mathbf{1}_{Z_{s+1} \in \partial\mathcal{D}_{s+1}} \omega \cdot (v_{s+1} - v_i) \times \\
& \quad \times \varphi_s(t, Z_s) f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1} dX_s dV_s dt
\end{aligned} \tag{33}$$

If $f_N(0) \in \mathcal{C}_0^\infty(\mathcal{D}_N)$ and $f_N(t)$ satisfies Liouville's equation, then the sequence of marginals $\{f_N^{(s)}(t)\}_{1 \leq s \leq N}$ solves the weak form of the BBGKY hierarchy. However, note that it is also possible to have solutions of the BBGKY hierarchy which are *not* sequences of marginals. Under suitable re-scalings, such solutions may have physical interpretations in the *grand canonical ensemble*, where the total number of particles is considered random. In our treatment, however, we will always be working in the *canonical ensemble*, since the total number of particles is just N .

We now turn to the dual BBGKY hierarchy. Given a pair of densities $F_N = \{f_N^{(s)}\}_{1 \leq s \leq N}$ and test functions $\Phi_N = \{\varphi_N^{(s)}\}_{1 \leq s \leq N}$, with each $f_N^{(s)}, \varphi_N^{(s)}$ symmetric under particle interchange, we define a duality bracket [17]:

$$\langle \Phi_N, F_N \rangle = \sum_{s=1}^N \frac{1}{s!} \int_{\mathcal{D}_s} \varphi_N^{(s)}(Z_s) f_N^{(s)}(Z_s) dZ_s \tag{34}$$

We would like to define the dual BBGKY hierarchy by the following duality relation:

$$\langle \Phi_N(t), F_N(0) \rangle = \langle \Phi_N(0), F_N(t) \rangle \tag{35}$$

which should hold whenever $F_N(t)$ solves the BBGKY hierarchy and $\Phi_N(t)$ solves the dual BBGKY hierarchy. Applying (35) and considering arbitrary weak solutions $F_N(t)$ of the BBGKY hierarchy, one can show that observables evolve according to the following hierarchy of equations (this is equivalent to equation 15 in [17], up to trivial re-scaling):

$$(\partial_t - V_s \cdot \nabla_{X_s}) \varphi_N^{(s)}(t, Z_s) = 0 \quad (Z_s \in \mathcal{D}_s, s = 1, \dots, N) \quad (36)$$

$$\begin{aligned} \frac{\varphi_N^{(s)}(t, Z_s^*)}{N-s+1} + \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) = \\ = \frac{\varphi_N^{(s)}(t, Z_s)}{N-s+1} + \varphi_N^{(s-1)}(t, Z_s^{(i)}) + \varphi_N^{(s-1)}(t, Z_s^{(j)}) \\ \left(Z_s \in \left(\Sigma_s(i, j) \times \mathbb{R}^{ds} \right) \cap \partial \mathcal{D}_s, s = 2, \dots, N \right) \end{aligned} \quad (37)$$

Given an initial data $\varphi_N^{(s)}(0)$, $1 \leq s \leq N$, we can solve this hierarchy recursively. The nonzero observable of lowest order (at the initial time, and therefore all time) simply evolves via the backwards Liouville flow. Once $\varphi_N^{(s-1)}(t)$ is known for all $t \geq 0$, it is possible to determine $\varphi_N^{(s)}(t)$ by integrating along characteristics. One uses the knowledge of $\varphi_N^{(s-1)}$ to determine the amount by which $\varphi_N^{(s)}$ “jumps” at particle collisions. Let us point out that as Z_s ranges over an open subset of $(\Sigma_s(i, j) \times \mathbb{R}^{ds}) \cap \partial \mathcal{D}_s$, the coordinates $Z_s^{(i)}, \dots$, cover an open subset of \mathcal{D}_{s-1} . Thus the source terms arising from $\varphi_N^{(s-1)}$ are always well-defined functions on the set $\partial \mathcal{D}_s$. Note that, by a density argument involving a Duhamel-type formula, it is possible to use initial data $\Phi_N(0)$ which does not satisfy the boundary condition (37).

5. LOCAL *a priori* BOUNDS ON OBSERVABLES

We will prove weighted \mathcal{L}^1 bounds on observables which are independent of N ; the stylized \mathcal{L} is intended to distinguish the spaces in which we bound observables. The proof is a dualization of the classical proof of *a priori* bounds on the marginals $f_N^{(s)}$ in weighted L^∞ spaces, originally due to Lanford. [15, 26] As in the case of Lanford’s theorem, the *a priori* bounds will only hold on a short time interval. Let us fix weight parameters $\beta > 0, \mu \in \mathbb{R}$, and define the norms

$$\|\Phi_N\|_{\mathcal{L}_{\beta, \mu}^1} = \sum_{s=1}^N \frac{1}{s!} \int_{\mathcal{D}_s} \left| \varphi_N^{(s)}(Z_s) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s \quad (38)$$

$$|F_N|_{L_{\beta, \mu}^\infty} = \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(Z_s) \right| e^{\beta E_s(Z_s)} e^{\mu s} \quad (39)$$

where $E_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |v_i|^2$. Then we have

$$\langle \Phi_N, F_N \rangle \leq \|\Phi_N\|_{\mathcal{L}_{\beta, \mu}^1} |F_N|_{L_{\beta, \mu}^\infty} \quad (40)$$

Since $\varphi_N^{(s)}$ is transported along characteristics within \mathcal{D}_s , $\left|\varphi_N^{(s)}(t, Z_s)\right|$ is transported as well. Therefore we have

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left|\varphi_N^{(s)}(t, Z_s)\right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s = \\
& = \int_{\mathcal{D}_s} V_s \cdot \nabla_{X_s} \left|\varphi_N^{(s)}(t, Z_s)\right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s \\
& = \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} n^{i,j} \cdot V_s \left|\varphi_N^{(s)}(t, Z_s)\right| e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
& = \frac{1}{2} \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} n^{i,j} \cdot V_s \times \\
& \quad \times \left(\left|\varphi_N^{(s)}(t, Z_s)\right| - \left|\varphi_N^{(s)}(t, Z_s^*)\right| \right) e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
& \leq \frac{1}{2} \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
& \quad \times \left| \varphi_N^{(s)}(t, Z_s) - \varphi_N^{(s)}(t, Z_s^*) \right| e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s
\end{aligned}$$

Now we employ the boundary condition to write

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left|\varphi_N^{(s)}(t, Z_s)\right| e^{-\beta E_s(Z_s)} e^{-\mu s} dZ_s \leq \\
& \leq \frac{N}{2} \sum_{1 \leq i < j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
& \quad \times \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) - \varphi_N^{(s-1)}(t, Z_s^{(i)}) - \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| \times \\
& \quad \times e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
& = \frac{N}{4} \sum_{i \neq j=1}^s \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
& \quad \times \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) + \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) - \varphi_N^{(s-1)}(t, Z_s^{(i)}) - \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| \times \\
& \quad \times e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
& \leq \frac{N}{4} \sum_{i \neq j=1}^s \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \left(\left|\varphi_N^{(s-1)}(t, (Z_s^*)^{(i)})\right| + \left|\varphi_N^{(s-1)}(t, (Z_s^*)^{(j)})\right| + \right. \\
& \quad \left. + \left|\varphi_N^{(s-1)}(t, Z_s^{(i)})\right| + \left|\varphi_N^{(s-1)}(t, Z_s^{(j)})\right| \right) e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s \\
& = N \sum_{i \neq j=1}^s \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \left|\varphi_N^{(s-1)}(t, Z_s^{(i)})\right| e^{-\beta E_s(Z_s)} e^{-\mu s} d\sigma^{i,j} dV_s
\end{aligned}$$

We can generalize this inequality to the case of time-dependent weights.

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \varphi_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
& \leq N \sum_{i \neq j=1}^s \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \times \\
& \quad \times \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} d\sigma^{i,j} dV_s + \\
& \quad + \int_{\mathcal{D}_s} \left| \varphi_N^{(s)}(t, Z_s) \right| \{ -\beta'(t)E_s(Z_s) - \mu'(t)s \} e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s
\end{aligned} \tag{41}$$

Note that in the case $s = 1$ the first term on the RHS vanishes (there are no source terms at the boundary).

Let us estimate just the first term. The integral over the hypersurface $\Sigma_s(i, j) = \{X_s \in \mathbb{R}^{ds} \mid |x_i - x_j| = \varepsilon\}$ brings down a factor of ε^{d-1} , which is then eliminated by virtue of the scaling $N\varepsilon^{d-1} = \ell^{-1}$.

$$\begin{aligned}
& N \sum_{i \neq j=1}^s \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} d\sigma^{i,j} dV_s \leq \\
& \leq \ell^{-1} \sum_{i=1}^s \int_{\mathbb{R}^{ds}} \int_{\mathbb{R}^{d(s-1)}} \int_{\mathbb{S}^{d-1}} \left(\sum_{j \neq i} |\omega \cdot (v_j - v_i)| \right) \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\
& \quad \times e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} d\omega dX_s^{(i)} dV_s \\
& \leq \ell^{-1} \sum_{i=1}^s \int_{\mathbb{R}^{ds}} \int_{\mathbb{R}^{d(s-1)}} \int_{\mathbb{S}^{d-1}} \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\
& \quad \times \left(\sqrt{2}(s-1)^{\frac{1}{2}} E_{s-1}(Z_s^{(i)})^{\frac{1}{2}} + (s-1)|v_i| \right) e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} d\omega dX_s^{(i)} dV_s \\
& \leq C_d \ell^{-1} e^{-\mu(t)} \beta(t)^{-\frac{d}{2}} s \int_{\mathbb{R}^{d(s-1)} \times \mathbb{R}^{d(s-1)}} \left| \varphi_N^{(s-1)}(t, Z_{s-1}) \right| \times \\
& \quad \times \left((s-1)^{\frac{1}{2}} E_{s-1}(Z_{s-1})^{\frac{1}{2}} + (s-1)\beta(t)^{-\frac{1}{2}} \right) \times \\
& \quad \times e^{-\beta(t)E_{s-1}(Z_{s-1})} e^{-\mu(t)(s-1)} dX_{s-1} dV_{s-1}
\end{aligned}$$

We may sum over s to obtain:

$$\begin{aligned}
& \frac{\partial}{\partial t} \|\Phi_N(t)\|_{\mathcal{L}_{\beta(t), \mu(t)}^1} \leq \\
& \leq \sum_{s=2}^N \frac{1}{s!} C_d \ell^{-1} e^{-\mu(t)} \beta(t)^{-\frac{d}{2}} s \int_{\mathcal{D}_{s-1}} \left| \varphi_N^{(s-1)}(t, Z_{s-1}) \right| \times \\
& \times \left((s-1)^{\frac{1}{2}} E_{s-1}(Z_{s-1})^{\frac{1}{2}} + \frac{(s-1)}{\beta(t)^{\frac{1}{2}}} \right) e^{-\beta(t) E_{s-1}(Z_{s-1})} e^{-\mu(t)(s-1)} dZ_{s-1} + \\
& + \sum_{s=1}^N \frac{1}{s!} \int_{\mathcal{D}_s} \left| \varphi_N^{(s)}(t, Z_s) \right| \left\{ -\beta'(t) E_s(Z_s) - \mu'(t)s \right\} e^{-\beta(t) E_s(Z_s)} e^{-\mu(t)s} dZ_s
\end{aligned} \tag{42}$$

We re-index the first term and combine; we furthermore assume that $\beta'(t), \mu'(t) > 0$ (this is *opposite* the usual convention because of duality). Then we have:

$$\begin{aligned}
& \frac{\partial}{\partial t} \|\Phi_N(t)\|_{\mathcal{L}_{\beta(t), \mu(t)}^1} \leq \\
& \leq \sum_{s=1}^{N-1} \frac{1}{s!} \int_{\mathcal{D}_s} \left| \varphi_N^{(s)}(t, Z_s) \right| \times \\
& \times \left[C_d \ell^{-1} e^{-\mu(t)} \beta(t)^{-\frac{d}{2}} \left(s^{\frac{1}{2}} E_s(Z_s)^{\frac{1}{2}} + s \beta(t)^{-\frac{1}{2}} \right) - \beta'(t) E_s(Z_s) - \mu'(t)s \right] \times \\
& \times e^{-\beta(t) E_s(Z_s)} e^{-\mu(t)s} dZ_s
\end{aligned} \tag{43}$$

It is now apparent that $\Phi_N(t)$ is controlled as long as the quantity in brackets is everywhere nonpositive, for $0 \leq t \leq T$ and $Z_s \in \mathcal{D}_s$. For instance, let us suppose that $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ are given. Then as long as $T_L > 0$ is chosen so that

$$T_L \leq C'_d \ell e^{\mu_0} \beta_0^{\frac{d+1}{2}} \tag{44}$$

then we shall have

$$\sup_{0 \leq t \leq T_L} \|\Phi_N(t)\|_{\mathcal{L}_{\beta_0, \mu_0}^1} \leq \|\Phi_N(0)\|_{\mathcal{L}_{\frac{1}{2}\beta_0, (\mu_0-1)}^1} \tag{45}$$

which implies by duality

$$\sup_{0 \leq t \leq T_L} |F_N(t)|_{L_{\frac{1}{2}\beta_0, (\mu_0-1)}^\infty} \leq |F_N(0)|_{L_{\beta_0, \mu_0}^\infty} \tag{46}$$

since the initial observable $\Phi_N(0)$ is arbitrary. Hence we obtain:

Theorem 5.1. *Suppose $F_N(t) = \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N}$ is a solution of the BBGKY hierarchy (33), subject to the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, and*

with each function $f_N^{(s)}(t, Z_s)$ symmetric under particle interchange. Further suppose that for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,

$$\sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1 \quad (47)$$

Then there is a constant $C_d > 0$, depending only on d , such that if $T_L < C_d \ell e^{\mu_0} \beta_0^{\frac{d+1}{2}}$ then there holds

$$\sup_{0 \leq t \leq T_L} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s)} e^{(\mu_0 - 1)s} \leq 1 \quad (48)$$

Remark. Theorem 5.1 does not require the functions $f_N^{(s)}$ to be non-negative, nor does it require that they form a consistent sequence of marginals.

The bound (46) is just the classical *a priori* bound of Lanford [15, 26]; note that the same argument based on observables would have worked in a periodic domain as well. Moreover, for any fixed initial datum, the Lanford time T_L increases in direct proportion to the mean free path, which is $\mathcal{O}(\ell)$.

6. GLOBAL *a priori* BOUNDS ON OBSERVABLES

Our goal is to extend the *a priori* bounds from the previous section to the entire time interval, $t \in [0, \infty)$, as soon as the mean free path $\mathcal{O}(\ell)$ is sufficiently large. The relevant estimates were first proved by Illner & Pulvirenti [22], using the dispersive inequalities we have stated in Lemmas 3.1, 3.2. Our approach is slightly different, in that we will be working with the *dual* hierarchy. Note that once the correct weights are chosen, the rest amounts to a computation, plus one application of Lemma 3.1.

Let us be given a time $T > 0$, and smooth increasing functions $\beta(t) : [0, T] \rightarrow \mathbb{R}^+$, $\mu(t) : [0, T] \rightarrow \mathbb{R}$. The spaces $\mathcal{L}_{\beta, \mu}^1$, $L_{\beta, \mu}^\infty$ are as defined in the previous section. We are given functions $\Phi_N(t) = \left\{ \varphi_N^{(s)}(t) \right\}_{1 \leq s \leq N}$, with each $\varphi_N^{(s)} : [0, T] \times \mathcal{D}_s \rightarrow \mathbb{R}$ symmetric under particle interchange, such that Φ_N satisfies the dual hierarchy (36-37) for $t \in [0, T]$. Define the functions $\tilde{\Phi}_N(t) = \left\{ \tilde{\varphi}_N^{(s)}(t) \right\}_{1 \leq s \leq N}$ by the formula

$$\tilde{\varphi}_N^{(s)}(t, Z_s) = \varphi_N^{(s)}(t, Z_s) e^{-\beta(t) I_s((X_s - (T-t)V_s, V_s))} \quad (49)$$

We will be estimating $\left\| \tilde{\Phi}_N(t) \right\|_{\mathcal{L}_{\beta(t), \mu(t)}^1}$ for $t \in [0, T]$.

Observe first that $(\partial_t - V_s \cdot \nabla_{X_s}) I_s((X_s - (T-t)V_s, V_s)) = 0$ on any open subset of \mathcal{D}_s . On the other hand, for $Z_s = (X_s, V_s) \in \mathcal{D}_s$ we have

$$I_s((X_s - (T-t)V_s, V_s)) = I_s(Z_s) - (T-t) \mathcal{Y}_s(Z_s) + (T-t)^2 E_s(Z_s) \quad (50)$$

Clearly if $Z_s \in \partial\mathcal{D}_s$ then $I_s(Z_s^*) = I_s(Z_s)$, and $E_s(Z_s^*) = E_s(Z_s)$. Hence by Lemma 3.1,

$$I_s((X_s - (T - t)V_s, V_s)) \geq I_s((X_s - (T - t)V_s^*, V_s^*)) \quad (51)$$

whenever $t \in [0, T]$ and $Z_s = (X_s, V_s) \in \partial\mathcal{D}_s$ is *pre-collisional*

The restriction $t \leq T$ in (51) is crucial; without this restriction the inequality could go the *wrong way* where we need it in the proof.

On any open subset of \mathcal{D}_s we have

$$\left(\frac{\partial}{\partial t} - V_s \cdot \nabla_{X_s} \right) \left| \varphi_N^{(s)}(t, Z_s) \right| = 0 \quad (52)$$

and likewise

$$\left(\frac{\partial}{\partial t} - V_s \cdot \nabla_{X_s} \right) I_s((X_s - (T - t)V_s, V_s)) = 0 \quad (53)$$

Therefore by the divergence theorem we obtain the *equality*:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s = \\ &= \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i, j)} n^{i, j} \cdot V_s \left| \varphi_N^{(s)}(t, Z_s) \right| \times \\ & \quad \times e^{-\beta(t)[I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i, j} dV_s + \\ &+ \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\ & \quad \times \{ -\beta'(t) [I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \} dZ_s \end{aligned} \quad (54)$$

The boundary term can be re-written as an integral over *pre-collisional* configurations. Recall that, according to our conventions, $n^{i, j} \cdot V_s = -\frac{x_j - x_i}{\varepsilon\sqrt{2}} \cdot (v_j - v_i)$ along $\Sigma_s(i, j) \times \mathbb{R}^{ds}$; therefore, $n^{i, j} \cdot V_s > 0$ for pre-collisional configurations. We have:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s = \\ &= \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i, j)} |n^{i, j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s) \right| \times \\ & \quad \times e^{-\beta(t)[I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i, j} dV_s \\ &- \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i, j)} |n^{i, j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s^*) \right| \times \\ & \quad \times e^{-\beta(t)[I_s((X_s - (T - t)V_s^*, V_s^*)) + E_s(Z_s^*)]} e^{-\mu(t)s} d\sigma^{i, j} dV_s \\ &+ \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\ & \quad \times \{ -\beta'(t) [I_s((X_s - (T - t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \} dZ_s \end{aligned} \quad (55)$$

According to the boundary condition (37), for any $Z_s \in \partial\mathcal{D}_s$,

$$\begin{aligned} \left| \varphi_N^{(s)}(t, Z_s) \right| &\leq \left| \varphi_N^{(s)}(t, Z_s^*) \right| + N \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| + N \left| \varphi_N^{(s-1)}(t, Z_s^{(j)}) \right| + \\ &\quad + N \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) \right| + N \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(j)}) \right| \end{aligned} \quad (56)$$

Therefore,

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\ &\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s^*) \right| \times \\ &\quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\ &+ N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\ &\quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\ &+ N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) \right| \times \\ &\quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\ &- \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s^*) \right| \times \\ &\quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s^*, V_s^*)) + E_s(Z_s^*)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\ &+ \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\ &\quad \times \left\{ -\beta'(t) [I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \right\} dZ_s \end{aligned} \quad (57)$$

We apply (51) to the first and third terms on the right hand side, for $0 \leq t \leq T$.

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
& \leq \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s^*) \right| \times \\
& \quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s^*, V_s^*)) + E_s(Z_s^*)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\
& + N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\
& \quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\
& + N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, (Z_s^*)^{(i)}) \right| \times \\
& \quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s^*, V_s^*)) + E_s(Z_s^*)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\
& - \frac{1}{2} \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s^{\text{inc}}(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s)}(t, Z_s^*) \right| \times \\
& \quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s^*, V_s^*)) + E_s(Z_s^*)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\
& + \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\
& \quad \times \{ -\beta'(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \} dZ_s
\end{aligned} \tag{58}$$

Now the first term precisely cancels the fourth term, whereas the second and third terms combine to yield an integral over all of $\Sigma_s(i, j)$.

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
& \leq N \sum_{1 \leq i \neq j \leq s} \int_{\mathbb{R}^{ds}} \int_{\Sigma_s(i,j)} |n^{i,j} \cdot V_s| \left| \varphi_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\
& \quad \times e^{-\beta(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)]} e^{-\mu(t)s} d\sigma^{i,j} dV_s \\
& + \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\
& \quad \times \{ -\beta'(t)[I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \} dZ_s
\end{aligned} \tag{59}$$

The following inequality is immediate and holds for all $Z_s \in \mathbb{R}^{2ds}$ and $t \in \mathbb{R}$:

$$\begin{aligned}
I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s) & \geq \\
& \geq \frac{1}{2} \left(|x_i - (T-t)v_i|^2 + |v_i|^2 \right) + E_{s-1}(Z_s^{(i)})
\end{aligned} \tag{60}$$

We may eliminate x_i from the right-hand side of (60) whenever $Z_s \in \Sigma_s(i, j) \times \mathbb{R}^{ds}$, due to the condition $x_j = x_i + \varepsilon\omega$. Combining this fact with the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, we obtain the following from (59):

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
& \leq \ell^{-1} \sum_{i=1}^s \int_{\mathbb{R}^{2d(s-1)}} \left| \tilde{\varphi}_N^{(s-1)}(t, Z_s^{(i)}) \right| \times \\
& \quad \times \left[\sum_{\substack{j=1 \\ j \neq i}}^s \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |\omega \cdot (v_j - v_i)| e^{-\frac{1}{2}\beta(t)[|x_j - \varepsilon\omega - (T-t)v_i|^2 + |v_i|^2]} d\omega dv_i \right] \times \\
& \quad \times e^{-\beta(t)E_{s-1}(Z_s^{(i)})} e^{-\mu(t)s} dZ_s^{(i)} \\
& + \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\
& \quad \times \left\{ -\beta'(t) [I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \right\} dZ_s
\end{aligned} \tag{61}$$

The integral in brackets is controlled using the classical dispersive inequality [3]:

$$\|\zeta(x - vt, v)\|_{L_x^\infty L_v^1} \leq |t|^{-d} \|\zeta(x, v)\|_{L_x^1 L_v^\infty} \tag{62}$$

Hence,

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} dZ_s \leq \\
& \leq \ell^{-1} s \int_{\mathbb{R}^{2d(s-1)}} \left| \tilde{\varphi}_N^{(s-1)}(t, Z_{s-1}) \right| \times \\
& \quad \times \left[C_d [1 + (T-t)]^{-d} \beta(t)^{-\frac{d}{2}} \left((s-1)^{\frac{1}{2}} E_{s-1}(Z_{s-1})^{\frac{1}{2}} + (s-1) \beta(t)^{-\frac{1}{2}} \right) \right] \times \\
& \quad \times e^{-\beta(t)E_{s-1}(Z_{s-1})} e^{-\mu(t)s} dZ_{s-1} \\
& + \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\
& \quad \times \left\{ -\beta'(t) [I_s((X_s - (T-t)V_s, V_s)) + E_s(Z_s)] - \mu'(t)s \right\} dZ_s
\end{aligned} \tag{63}$$

We can sum over s to obtain, for $0 \leq t \leq T$,

$$\begin{aligned} & \frac{\partial}{\partial t} \left\| \tilde{\Phi}_N(t) \right\|_{\mathcal{L}_{\beta(t), \mu(t)}^1} \leq \\ & \leq \sum_{s=1}^{N-1} \frac{1}{s!} \int_{\mathcal{D}_s} \left| \tilde{\varphi}_N^{(s)}(t, Z_s) \right| e^{-\beta(t)E_s(Z_s)} e^{-\mu(t)s} \times \\ & \times \left[\frac{C_d e^{-\mu(t)} \beta(t)^{-\frac{d}{2}}}{\ell [1 + (T-t)]^d} \left(s^{\frac{1}{2}} E_s(Z_s)^{\frac{1}{2}} + s \beta(t)^{-\frac{1}{2}} \right) - \beta'(t) E_s(Z_s) - \mu'(t)s \right] dZ_s \end{aligned} \quad (64)$$

Suppose $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$ are given. Then fixing any $T > 0$ we define

$$\beta(t) = \beta_0 - \frac{1}{2} \beta_0 \left(1 - [1 + (T-t)]^{-(d-1)} \right) \quad (65)$$

$$\mu(t) = \mu_0 - \left(1 - [1 + (T-t)]^{-(d-1)} \right) \quad (66)$$

We have $\beta(T) = \beta_0$, $\mu(T) = \mu_0$, $\inf_{0 \leq t \leq T} \beta(t) \geq \frac{1}{2} \beta_0$, $\inf_{0 \leq t \leq T} \mu(t) \geq (\mu_0 - 1)$, and

$$\beta'(t) = \frac{1}{2} \beta_0 (d-1) [1 + (T-t)]^{-d} \quad (67)$$

$$\mu'(t) = (d-1) [1 + (T-t)]^{-d} \quad (68)$$

Then if we assume further that $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small (depending only on d), then

$$\sup_{0 \leq t \leq T} \left\| \tilde{\Phi}_N(t) \right\|_{\mathcal{L}_{\beta(t), \mu(t)}^1} \leq \left\| \tilde{\Phi}_N(0) \right\|_{\mathcal{L}_{\frac{1}{2}\beta_0, (\mu_0-1)}^1} \quad (69)$$

In particular,

$$\left\| \tilde{\Phi}_N(T) \right\|_{\mathcal{L}_{\beta_0, \mu_0}^1} \leq \left\| \tilde{\Phi}_N(0) \right\|_{\mathcal{L}_{\frac{1}{2}\beta_0, (\mu_0-1)}^1} \quad (70)$$

Since $T > 0$ is arbitrary, recalling the definition of $\tilde{\Phi}_N$ and using duality we obtain:

Theorem 6.1. (*Illner & Pulvirenti 1989*) Suppose $F_N(t) = \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N}$ is a solution of the BBGKY hierarchy (33), subject to the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, and with each function $f_N^{(s)} : [0, \infty) \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ symmetric under particle interchange. Further suppose that for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,

$$\sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 [E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \leq 1 \quad (71)$$

Then if $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small (depending only on d) then we have

$$\sup_{t \geq 0} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2}\beta_0 [E_s(Z_s) + I_s((X_s - V_s t, V_s))]} e^{(\mu_0 - 1)s} \leq 1 \quad (72)$$

Remark. Theorem 6.1 does not require the functions $f_N^{(s)}$ to be non-negative, nor does it require that they form a consistent sequence of marginals.

7. REPRESENTATION OF MARGINALS VIA PSEUDO-TRAJECTORIES

We recall that any solution $f_N^{(s)}(t)$ of the BBGKY hierarchy may be decomposed in terms of the initial data propagated along “pseudo-trajectories.” This technique is first due to Lanford, and is now a standard tool in the analysis of the Boltzmann-Grad limit for hard spheres. To begin, observe that if $\{f_N^{(s)}(t, Z_s)\}_{1 \leq s \leq N}$ solves (33), then by considering test functions which vanish along $[0, \infty) \times \partial\mathcal{D}_s$, it follows that the densities $f_N^{(s)}$ solve the following equation in the sense of distributions:

$$\left(\frac{\partial}{\partial t} + V_s \cdot \nabla_{X_s}\right) f_N^{(s)}(t, Z_s) = (N-s)\varepsilon^{d-1} C_{s+1} f_N^{(s+1)}(t, Z_s) \quad (73)$$

where $f_N^{(s)}(t, Z_s) = f_N^{(s)}(t, Z_s^*)$ a.e. $(t, Z_s) \in [0, \infty) \times \partial\mathcal{D}_s$, and C_{s+1} is the collision operator

$$C_{s+1} = \sum_{i=1}^s C_{i,s+1} \quad (74)$$

$$C_{i,s+1} = C_{i,s+1}^+ - C_{i,s+1}^- \quad (75)$$

$$\begin{aligned} C_{i,s+1}^+ f_N^{(s+1)}(t, Z_s) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{Z_{s+1} \in \partial\mathcal{D}_{s+1}} [\omega \cdot (v_{s+1} - v_i)]_+ \times \\ &\quad \times f_N^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i + \varepsilon\omega, v_{s+1}^*) d\omega dv_{s+1} \end{aligned} \quad (76)$$

$$\begin{aligned} C_{i,s+1}^- f_N^{(s+1)}(t, Z_s) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{Z_{s+1} \in \partial\mathcal{D}_{s+1}} [\omega \cdot (v_{s+1} - v_i)]_- \times \\ &\quad \times f_N^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i, \dots, x_s, v_s, x_i + \varepsilon\omega, v_{s+1}) d\omega dv_{s+1} \end{aligned} \quad (77)$$

where

$$\begin{cases} v_i^* = v_i + \omega\omega \cdot (v_j - v_i) \\ v_j^* = v_j - \omega\omega \cdot (v_j - v_i) \end{cases} \quad (78)$$

We can re-write (73) by means of Duhamel’s formula, using the transport operator $T_s(t)$ defined by $(T_s(t)g_s)(Z_s) = g_s(\psi_s^{-t}Z_s)$ for any $g_s \in L^1(\mathcal{D}_s)$. The operators $T_s(t)$ form a strongly continuous semigroup on $L^1(\mathcal{D}_s)$, with generator given by $-V_s \cdot \nabla_{X_s}$ and specular reflection at the boundary $\partial\mathcal{D}_s$. We have

$$f_N^{(s)}(t) = T_s(t)f_N^{(s)}(0) + (N-s)\varepsilon^{d-1} \int_0^t T_s(t-t_1)C_{s+1}f_N^{(s+1)}(t_1)dt_1 \quad (79)$$

Now by iterating this formula we can write the marginal $f_N^{(s)}(t)$ as a *finite* sum of terms, each of which depends only on the initial data:

$$f_N^{(s)}(t) = \sum_{k=0}^{N-s} a_{N,k,s} \times \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} T_s(t-t_1) C_{s+1} \dots T_{s+k}(t_k) f_N^{(s+k)}(0) dt_k \dots dt_1 \quad (80)$$

where

$$a_{N,k,s} = \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \quad (81)$$

Since we enforce the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, we have $0 \leq a_{N,k,s} \leq \ell^{-k}$ and $a_{N,k,s}\ell^k \rightarrow 1$ as $N \rightarrow \infty$ with k, s fixed.

The Duhamel series (80) may be interpreted as a way of describing the solution $F_N(t)$ in terms of the data $F_N(0)$ propagated along a family of artificial trajectories, or “pseudo-trajectories.” [15, 26, 32] Given $Z_s \in \mathcal{D}_s$, along with times $0 \leq t_k \leq \dots \leq t_1 \leq t$, velocities v_{s+1}, \dots, v_{s+k} , impact parameters $\omega_1, \dots, \omega_k$, and indices i_1, \dots, i_k , we will define

$$Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \in \mathcal{D}_{s+k} \quad (82)$$

We assume $i_1 \in \{1, \dots, s\}$, $i_2 \in \{1, \dots, s, s+1\}$, \dots , $i_j \in \{1, 2, \dots, s+j-1\}$; we will also need to assume that certain “exclusion conditions” are satisfied, as will become clear. To begin the induction, for $Z_s \in \mathcal{D}_s$ and $t > 0$ we define

$$Z_{s,s} [Z_s, t] = \psi_s^{-t} Z_s \quad (83)$$

More generally, if the symbol

$$Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \in \mathcal{D}_{s+k} \quad (84)$$

is defined, then for $\tau > 0$ we define

$$\begin{aligned} Z_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] &= \\ &= \psi_{s+k}^{-\tau} Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \end{aligned} \quad (85)$$

Similarly, if the symbol

$$\begin{aligned} Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] &= \\ &= (X'_{s+k}, V'_{s+k}) \in \mathcal{D}_{s+k} \end{aligned} \quad (86)$$

is defined (including the possibility $k = 0$) then for any given velocity $v_{s+k+1} \in \mathbb{R}^d$, any index $i_{k+1} \in \{1, \dots, s, s+1, \dots, s+k\}$, and any “suitable” choice of impact parameter $\omega_{k+1} \in \mathbb{S}^{d-1}$, if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0$

then we define

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] = \\ = \left(x'_1, v'_1, \dots, x'_{i_{k+1}}, v'_{i_{k+1}}, \dots, x'_s, v'_s, x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1} \right) \end{aligned} \quad (87)$$

whereas if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0$ then we define

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] = \\ = \left(x'_1, v'_1, \dots, x'_{i_{k+1}}, v'_{i_{k+1}} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}), \right. \\ \left. \dots, x'_s, v'_s, x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \right) \end{aligned} \quad (88)$$

Here a “suitable” impact parameter ω is one for which $|x'_{i_{k+1}} + \varepsilon \omega - x'_j| > \varepsilon$ for each $j \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}$; note that the set of suitable impact parameters may be empty.

Remark. The physical interpretation of the above construction is that s particles begin in configuration $Z_s \in \mathcal{D}_s$ at time t , then evolve under the *backwards* hard sphere flow for a time $t - t_1$; at time t_1 , the $(s+1)$ st particle is created adjacent to the i_1 st particle with velocity v_{s+1} . If the pair $(i_1, s+1)$ is in a post-collisional configuration, then we perform an instantaneous collision to place the particles in a pre-collisional configuration. To continue the flow, we push the system through the backwards flow of $(s+1)$ hard spheres for a time $t_1 - t_2$, and so forth. The state of the process at time 0 is then $Z_{s,s+k} [Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k]$.

Remark. As a matter of convenience, we have enforced a convention whereby particles are always in a pre-collisional configuration at the moment that a new particle is created. Keep in mind, however, that the backwards flow can subsequently place particles into a post-collisional configuration, though this can only happen between particle creations.

We will also require an iterated collision kernel

$$b_{s,s+k} [Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k] \quad (89)$$

in order to account for each added particle. First we define

$$b_{s,s} [Z_s, t] = \mathbf{1}_{Z_s \in \mathcal{D}_s} \quad (90)$$

If we have defined

$$b_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \quad (91)$$

then there are two cases: (i) $Z_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}) \in \mathcal{D}_{s+k}$ is well-defined as above, in which case

$$\begin{aligned} b_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] &= \\ &= b_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \end{aligned} \quad (92)$$

$$\begin{aligned} b_{s,s+k+1} [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] &= \\ = \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \times \\ \times \left(\prod_{j \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} \mathbf{1}_{|x'_{i_{k+1}} + \varepsilon \omega_{k+1} - x'_j| > \varepsilon} \right) \times \\ \times b_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \end{aligned} \quad (93)$$

(ii) otherwise,

$$\begin{aligned} b_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] &= \\ = b_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] &= 0 \end{aligned} \quad (94)$$

$$\begin{aligned} b_{s,s+k+1} [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] &= 0 \end{aligned} \quad (95)$$

Then the Duhamel series (80) becomes

$$\begin{aligned} f_N^{(s)}(t, Z_s) &= \sum_{k=0}^{N-s} a_{N,k,s} \times \\ &\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \\ &\times \left(b_{s,s+k} [\cdot] f_N^{(s+k)}(0, Z_{s,s+k} [\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned} \quad (96)$$

Remark. The collision kernel $b_{s,s+k}[\dots]$ vanishes automatically whenever $Z_{s,s+k}[\dots]$ fails to be well-defined. This convention is convenient because it allows us to specify a fixed N -independent domain of integration in (96).

8. STABILITY OF PSEUDO-TRAJECTORIES

The purpose of this section is to prove that typical pseudo-trajectories are stable with respect to the creation of a new particle, at least outside a small set of creation times, velocities, and impact parameters. The main novelty of this stability result, compared to previous results [15], is that we are able to allow particles to pass arbitrarily close to each other in *space* under the backwards flow, as long as they do not collide. The price we pay for this improvement is that we must make explicit use of the time integrals

appearing in the Duhamel series (96), and employ an unusual cut-off for nearby velocities. This proof is inspired in part by the ideas from [32]; note, however, that there the authors required more sophisticated cut-offs to deal with rather general physical interactions.

We will require the following elementary geometrical fact (the proof is trivial):

Lemma 8.1. *Fix $v \in \mathbb{R}^d \setminus \{0\}$, and for $\omega \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ (where \mathbb{S}^{d-1} is the unit sphere centered on the origin) define*

$$u_\omega = |v|^{-1} (2\omega \omega \cdot v - v) \quad (97)$$

then $u_\omega \in \mathbb{S}^{d-1}$ for each $\omega \in \mathbb{S}^{d-1}$. If $\mathbb{S}_v^{d-1} = \{\omega \in \mathbb{S}^{d-1} \mid \omega \cdot v > 0\}$ then the map $\omega \mapsto u_\omega$ restricts to a diffeomorphism $\mathbb{S}_v^{d-1} \rightarrow \mathbb{S}^{d-1} \setminus \{-|v|^{-1}v\}$.

We will also need:

Lemma 8.2. *Let $L \subset \mathbb{R}^d$ ($d \geq 2$) be a line, and for $\rho > 0$ consider the solid cylinder*

$$\mathcal{C}_\rho = \left\{ u \in \mathbb{R}^d \mid \text{dist}(u, L) \leq \rho \right\} \quad (98)$$

Then

$$\int_{\mathbb{S}^{d-1}} \mathbf{1}_{\omega \in \mathcal{C}_\rho} d\omega \leq C_d \rho^{(d-1)/2} \quad (99)$$

where the constant C_d does not depend on the choice of line L .

Proof. There are two cases: either \mathcal{C}_ρ contains a point which is within distance $1 - 3\rho$ of the sphere's center, or it does not. In the first case, we clearly have

$$\int_{\mathbb{S}^{d-1}} \mathbf{1}_{\omega \in \mathcal{C}_\rho} d\omega \leq C_d \rho^{d-\frac{3}{2}}$$

In the second case, we can estimate the size of a spherical cap to obtain that

$$\int_{\mathbb{S}^{d-1}} \mathbf{1}_{\omega \in \mathcal{C}_\rho} d\omega \leq C_d \rho^{(d-1)/2}$$

Since $d \geq 2$, we can take the maximum of these two bounds to obtain (99). \square \square

We now turn to the main result for this section. To state the proposition, we must fix a parameter $\eta > 0$ and introduce the following sets:

$$\mathcal{K}_s = \left\{ Z_s = (X_s, V_s) \in \overline{\mathcal{D}_s} \mid \psi_s^{-\tau} Z_s = (X_s - V_s \tau, V_s) \ \forall \ \tau > 0 \right\} \quad (100)$$

$$\mathcal{U}_s^\eta = \left\{ Z_s = (X_s, V_s) \in \overline{\mathcal{D}_s} \mid \inf_{1 \leq i < j \leq s} |v_i - v_j| > \eta \right\} \quad (101)$$

Remark. The condition $Z_s \in \mathcal{U}_s^\eta$ is meant to force particles to disperse away from each other under the action of the free flow.

Proposition 8.3. *There is a constant $c_d > 0$, depending only on the dimension d , such that all the following holds: Assume that*

$$\begin{aligned} Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = (X'_{s+k}, V'_{s+k}) \in \mathcal{K}_{s+k} \cap \mathcal{U}_{s+k}^\eta \end{aligned} \quad (102)$$

and $E_{s+k} (Z'_{s+k}) \leq 2R^2$; then,

(i) for all $\tau \geq 0$ we have

$$\begin{aligned} Z_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \\ \in \mathcal{K}_{s+k} \cap \mathcal{U}_{s+k}^\eta \end{aligned} \quad (103)$$

(ii) for any $i_{k+1} \in \{1, \dots, s, s+1, \dots, s+k\}$, and for any $\theta, \alpha, y > 0$ such that $\sin \theta > c_d y^{-1} \varepsilon$, there exists a measurable set $\mathcal{B} \subset [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, which may depend on Z_s, t , and $\{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k$, such that

$$\forall \eta < R, \forall T > 0,$$

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_d (s+k) T R^d \left[\alpha + \frac{y}{\eta T} + C_{d,\alpha} \left(\frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right] \end{aligned} \quad (104)$$

and

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] \\ \in \mathcal{K}_{s+k+1} \cap \mathcal{U}_{s+k+1}^\eta \end{aligned} \quad (105)$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \setminus \mathcal{B}$.

Proof. Claim (i) is trivial. For claim (ii), we distinguish between two possibilities for the added particle: either $(\tau, v_{s+k+1}, \omega_{k+1})$ is such that $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0$, or else $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0$. We introduce two sets,

$$\mathcal{A}^+ = \left\{ \begin{array}{l} (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that} \\ \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0 \end{array} \right\} \quad (106)$$

$$\mathcal{A}^- = \left\{ \begin{array}{l} (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that} \\ \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0 \end{array} \right\} \quad (107)$$

then we write $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ where $\mathcal{B}^+ \subset \mathcal{A}^+$ and $\mathcal{B}^- \subset \mathcal{A}^-$.

Construction of \mathcal{B}^- . We first eliminate creation times τ which could result in spatial concentrations of particles. This is where we use the property that $Z'_{s+k} \in \mathcal{U}_{s+k}^\eta$, since this condition guarantees that two particles can only be

close to each other for a short time (as long as the $(s+k)$ particles evolve under the free flow). We introduce the set

$$\mathcal{B}_I^- = \left\{ \begin{array}{l} (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that} \\ \inf_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} \left| (x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right| \leq y \end{array} \right\} \quad (108)$$

then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_I^-} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_d (s+k-1) R^d \eta^{-1} y \end{aligned} \quad (109)$$

As a technical matter, we must also guarantee that the $(s+k+1)$ -particle state lives in \mathcal{U}_{s+k+1}^η at the time of particle creation. Hence, we will define

$$\mathcal{B}_{II}^- = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \mid \inf_{1 \leq i \leq s+k} |v_{s+k+1} - v'_i| \leq \eta \right\} \quad (110)$$

then we have

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{II}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d (s+k) T \eta^d \quad (111)$$

Lastly, we will guarantee (with high probability) that the created particle does not “recollide” under the backwards flow; that is, the $(s+k+1)$ -particle state must live in \mathcal{K}_{s+k+1} at the time of particle creation. To this end, for $i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}$ we introduce the set

$$\mathcal{B}_{III,i}^- = \left\{ \begin{array}{l} (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that} \\ \frac{\left| \left((x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right) \cdot (v_{s+k+1} - v'_i) \right|}{\left| (x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right| |v_{s+k+1} - v'_i|} \geq \cos \theta \end{array} \right\} \quad (112)$$

and we let $\mathcal{B}_{III}^- = \bigcup_{i \in \{1, \dots, s+k\} \setminus \{i_{k+1}\}} \mathcal{B}_{III,i}^-$; then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{III}^-} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_d (s+k-1) T R^d \theta^{d-1} \end{aligned} \quad (113)$$

Remark. The vector

$$(x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i)$$

is just the relative displacement between the i_{k+1} st particle and the i th particle at the time of the particle creation. On the other hand, $(v_{s+k+1} - v'_i)$ is the relative velocity between the $(s+k+1)$ st particle and the i th particle at the time of particle creation. Note that the $(s+k+1)$ st particle is created at a distance of ε from the i_{k+1} st particle. Hence the formula defining $\mathcal{B}_{III,i}^-$ is a “cone condition” whose complementary event prevents the newly created

$(s+k+1)$ st particle from colliding with the i th particle under the backwards hard sphere flow, as long as θ is not too small.

To conclude, we let $\mathcal{B}^- = \mathcal{B}_I^- \cup \mathcal{B}_{II}^- \cup \mathcal{B}_{III}^-$; then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_d (s+k) T R^d \left[\frac{y}{\eta T} + \left(\frac{\eta}{R} \right)^d + \theta^{d-1} \right] \end{aligned} \quad (114)$$

Then again, by assumption, $\sin \theta > c_d y^{-1} \varepsilon$; by choosing c_d sufficiently large we may guarantee that

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] \\ \in \mathcal{K}_{s+k+1} \cap \mathcal{U}_{s+k+1}^\eta \end{aligned} \quad (115)$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus \mathcal{B}^-$.

Construction of \mathcal{B}^+ . The construction of \mathcal{B}^+ is very similar to the construction of \mathcal{B}^- ; the main difference is that we have to account for the change of variables arising from one collision. We will find it helpful to define the following notation:

$$v_{s+k+1}^* = v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \quad (116)$$

$$v_{i_{k+1}}'^* = v'_{i_{k+1}} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \quad (117)$$

Note that Z'_{s+k} is *fixed* as in the statement of the proposition, whereas $(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+$ are considered free parameters.

We eliminate creation times τ for which particles are too concentrated in space:

$$\mathcal{B}_I^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \text{ such that } \inf_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} \left| (x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right| \leq y \right\} \quad (118)$$

then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_I^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_d (s+k-1) R^d \eta^{-1} y \end{aligned} \quad (119)$$

We find it convenient to eliminate collisions which are too close to grazing; therefore, we define

$$\mathcal{B}_{II}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \text{ such that } \left| \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \right| \leq (\sin \alpha) \left| v_{s+k+1} - v'_{i_{k+1}} \right| \right\} \quad (120)$$

then we have

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{II}^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d T R^d \alpha \quad (121)$$

We introduce the next three sets to guarantee that the $(s+k+1)$ -particle state lives in \mathcal{U}_{s+k+1}^η . In this instance we must impose *multiple* conditions, since both the $(s+k+1)$ st particle and the i_{k+1} st particle are modified by the collision. Note that $|v_{s+k+1}^* - v_{i_{k+1}}'^*| = |v_{s+k+1} - v_{i_{k+1}}'|$.

$$\mathcal{B}_{III}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II}^+ \text{ such that } \inf_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} |v_{s+k+1}^* - v_i'| \leq \eta \right\} \quad (122)$$

$$\mathcal{B}_{IV}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II}^+ \text{ such that } \inf_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} |v_{i_{k+1}}'^* - v_i'| \leq \eta \right\} \quad (123)$$

$$\mathcal{B}_V^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \mid |v_{s+k+1} - v_{i_{k+1}}'| \leq \eta \right\} \quad (124)$$

Then using Lemma 8.1 and the definition of \mathcal{B}_{II}^+ , we obtain:

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{III}^+} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_{d,\alpha} (s+k-1) T R \eta^{d-1} \end{aligned} \quad (125)$$

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{IV}^+} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_{d,\alpha} (s+k-1) T R \eta^{d-1} \end{aligned} \quad (126)$$

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_V^+} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d T \eta^d \quad (127)$$

We will now show that, with high probability, the particle creation yields an $(s+k+1)$ -particle state in \mathcal{K}_{s+k+1} , hence the backwards hard sphere flow coincides with the free flow. For $i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}$, we define

$$\mathcal{B}_{VI,i}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II}^+ \text{ such that } \frac{\left| \left((x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right) \cdot (v_{s+k+1}^* - v'_i) \right|}{\left| (x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right| |v_{s+k+1}^* - v'_i|} \geq \cos \theta \right\} \quad (128)$$

$$\mathcal{B}_{VII,i}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}_{II}^+ \text{ such that } \frac{\left| \left((x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right) \cdot (v_{i_{k+1}}'^* - v'_i) \right|}{\left| (x'_{i_{k+1}} - x'_i) - \tau (v'_{i_{k+1}} - v'_i) \right| |v_{i_{k+1}}'^* - v'_i|} \geq \cos \theta \right\} \quad (129)$$

$$\mathcal{B}_{VI}^+ = \bigcup_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} \mathcal{B}_{VI,i}^+ \quad (130)$$

$$\mathcal{B}_{VII}^+ = \bigcup_{i \in \{1, \dots, s, s+1, \dots, s+k\} \setminus \{i_{k+1}\}} \mathcal{B}_{VII,i}^+ \quad (131)$$

Then using Lemmas 8.1 and 8.2, and the definition of \mathcal{B}_{II}^+ , we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{VI}^+} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_{d,\alpha} (s+k-1) TR^d \theta^{(d-1)/2} \end{aligned} \quad (132)$$

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{VII}^+} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_{d,\alpha} (s+k-1) TR^d \theta^{(d-1)/2} \end{aligned} \quad (133)$$

To conclude, we let $\mathcal{B}^+ = \mathcal{B}_I^+ \cup \mathcal{B}_{II}^+ \cup \mathcal{B}_{III}^+ \cup \mathcal{B}_{IV}^+ \cup \mathcal{B}_V^+ \cup \mathcal{B}_{VI}^+ \cup \mathcal{B}_{VII}^+$; then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^+} d\omega_{k+1} dv_{s+k+1} d\tau &\leq \\ &\leq C_d (s+k) TR^d \left[\alpha + \frac{y}{\eta T} + C_{d,\alpha} \left(\frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right] \end{aligned} \quad (134)$$

Then again, by assumption, we have $\sin \theta > c_d y^{-1} \varepsilon$; as long as c_d is chosen sufficiently large, we always have

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] \\ \in \mathcal{K}_{s+k+1} \cap \mathcal{U}_{s+k+1}^\eta \end{aligned} \quad (135)$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^+ \setminus \mathcal{B}^+$. □ □

9. THE BOLTZMANN HIERARCHY

We will say that a sequence of continuous symmetric functions $\left\{ f_\infty^{(s)}(t, Z_s) \right\}_{s \in \mathbb{N}}$, with $Z_s \in \mathbb{R}^{2ds}$, satisfies the Boltzmann hierarchy if the following equation holds for each s in the sense of distributions:

$$\left(\frac{\partial}{\partial t} + V_s \cdot \nabla_{X_s} \right) f_\infty^{(s)}(t, Z_s) = \ell^{-1} C_{s+1}^0 f_\infty^{(s+1)}(t, Z_s) \quad (136)$$

The collision operators $C_{s,s+1}^0$ are defined as follows:

$$C_{s+1}^0 = \sum_{i=1}^s C_{i,s+1}^0 \quad (137)$$

$$C_{i,s+1}^0 = C_{i,s+1}^{0,+} - C_{i,s+1}^{0,-} \quad (138)$$

$$C_{i,s+1}^{0,+} f_{\infty}^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\omega \cdot (v_{s+1} - v_i)]_+ \times \\ \times f_{\infty}^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) d\omega dv_{s+1} \quad (139)$$

$$C_{i,s+1}^{0,-} f_{\infty}^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\omega \cdot (v_{s+1} - v_i)]_- \times \\ \times f_{\infty}^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i, \dots, x_s, v_s, x_i, v_{s+1}) d\omega dv_{s+1} \quad (140)$$

where

$$\begin{cases} v_i^* = v_i + \omega \omega \cdot (v_j - v_i) \\ v_j^* = v_j - \omega \omega \cdot (v_j - v_i) \end{cases} \quad (141)$$

We also define the free transport operators $T_s^0(t)$, which act on functions $f_{\infty}^{(s)} : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ as follows:

$$(T_s^0(t) f_{\infty}^{(s)})(X_s, V_s) = f_{\infty}^{(s)}(X_s - V_s t, V_s) \quad (142)$$

Just as for the BBGKY hierarchy, the Boltzmann hierarchy admits a *formal* Duhamel series expressing the solution in terms of the data,

$$f_{\infty}^{(s)}(t) = \sum_{k=0}^{\infty} \ell^{-k} \\ \times \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} T_s^0(t - t_1) C_{s+1}^0 \dots T_{s+k}^0(t_k) f_{\infty}^{(s+k)}(0) dt_k \dots dt_1 \quad (143)$$

The convergence of this series (for small data) follows from the well-posedness theorem which is proven in the following section.

Remark. If $f_t(x, v)$ is a sufficiently smooth solution of the Boltzmann equation then the sequence $\{f_t^{\otimes s}\}_{s \in \mathbb{N}}$ is a solution of the Boltzmann hierarchy.

We will now construct psuedo-trajectories for the Boltzmann hierarchy, directly analogous to those we have constructed for the BBGKY hierarchy. [15, 26, 32] Given $Z_s \in \mathbb{R}^{2ds}$, along with times $0 \leq t_k \leq \dots \leq t_1 \leq t$, velocities v_{s+1}, \dots, v_{s+k} , impact parameters $\omega_1, \dots, \omega_k$, and indices i_1, \dots, i_k , we will define

$$Z_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \in \mathbb{R}^{2d(s+k)} \quad (144)$$

We assume $i_1 \in \{1, \dots, s\}$, $i_2 \in \{1, \dots, s, s+1\}$, \dots , $i_j \in \{1, 2, \dots, s+j-1\}$. To begin the induction, for $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ and $t > 0$ we define

$$Z_{s,s}^0 [Z_s, t] = (X_s - V_s t, V_s) \quad (145)$$

More generally, if the symbol

$$Z_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = (X'_{s+k}, V'_{s+k}) \in \mathbb{R}^{2d(s+k)} \quad (146)$$

is defined, then for $\tau > 0$ we define

$$\begin{aligned} Z_{s,s+k}^0 [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = (X'_{s+k} - V'_{s+k}\tau, V'_{s+k}) \end{aligned} \quad (147)$$

Similarly, if the symbol

$$\begin{aligned} Z_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = (X'_{s+k}, V'_{s+k}) \in \mathbb{R}^{2d(s+k)} \end{aligned} \quad (148)$$

is defined (including the possibility $k = 0$) then for any given velocity $v_{s+k+1} \in \mathbb{R}^d$, any index $i_{k+1} \in \{1, \dots, s, s+1, \dots, s+k\}$, and any choice of impact parameter $\omega_{k+1} \in \mathbb{S}^{d-1}$, if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0$ we define

$$\begin{aligned} Z_{s,s+k+1}^0 [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] = \\ = (x'_1, v'_1, \dots, x'_{i_{k+1}}, v'_{i_{k+1}}, \dots, x'_s, v'_s, x'_{i_{k+1}}, v_{s+k+1}) \end{aligned} \quad (149)$$

whereas if $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0$ then we define

$$\begin{aligned} Z_{s,s+k+1}^0 [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] = \\ = (x'_1, v'_1, \dots, x'_{i_{k+1}}, v'_{i_{k+1}} + \omega_{k+1}\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}), \\ \dots, x'_s, v'_s, x'_{i_{k+1}}, v_{s+k+1} - \omega_{k+1}\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}})) \end{aligned} \quad (150)$$

Now we construct the collision kernel $b_{s,s+k}^0 [Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k]$. First we define

$$b_{s,s}^0 [Z_s, t] = 1 \quad (151)$$

If we have defined

$$b_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \quad (152)$$

then for any $\tau > 0$ we define

$$\begin{aligned} b_{s,s+k}^0 [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = b_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \end{aligned} \quad (153)$$

and we also define

$$\begin{aligned} b_{s,s+k+1}^0 [Z_s, t; t_1, \dots, t_k, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] = \\ = \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \times \\ \times b_{s,s+k}^0 [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \end{aligned} \quad (154)$$

Then the formal Duhamel series (143) becomes

$$\begin{aligned}
f_\infty^{(s)}(t, Z_s) &= \sum_{k=0}^{\infty} \ell^{-k} \times \\
&\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \\
&\times \left(b_{s,s+k}^0 [\cdot] f_\infty^{(s+k)}(0, Z_{s,s+k}^0 [\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right]
\end{aligned} \tag{155}$$

10. SMALL SOLUTIONS OF THE BOLTZMANN HIERARCHY

We will prove a global well-posedness result for the Boltzmann hierarchy with small data $F_\infty(0) = \{f_\infty^{(s)}(0)\}_{s \in \mathbb{N}}$ in vacuum. The proof is based on a fixed point iteration and a dispersive estimate. [3, 21] If, in addition to the hypotheses of the theorem, we have $f_\infty^{(s)}(0) = f_0^{\otimes s}$ for some smooth function $f_0(x, v)$, then it is well-known that the Boltzmann equation has a unique non-negative smooth solution f_t [7, 11], and $\{f_t^{\otimes s}\}_{s \in \mathbb{N}}$ solves the Boltzmann hierarchy. Then the uniqueness part of the following theorem implies that $F_\infty(t) = \{f_t^{\otimes s}\}_{s \in \mathbb{N}}$, i.e., the Boltzmann hierarchy propagates chaoticity.

Theorem 10.1. (*Illner & Pulvirenti 1986*) Suppose $F_\infty(0) = \{f_\infty^{(s)}(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f_\infty^{(s)}(0) : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ is continuous and symmetric, and for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$,

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(0, Z_s) \right| e^{\beta_0[E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \leq 1 \tag{156}$$

Then if $d \geq 3$ and $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small (depending only on d), then there exists a unique sequence $F_\infty(t) = \{f_\infty^{(s)}(t)\}_{s \in \mathbb{N}}$, with each $f_\infty^{(s)}(t, Z_s) : [0, \infty) \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ continuous and symmetric, such that

$$\sup_{t \geq 0} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(t, Z_s) \right| e^{\frac{1}{2}\beta_0[E_s(Z_s) + I_s((X_s - V_s t, V_s))]} e^{(\mu_0 - 1)s} \leq 2 \tag{157}$$

and for each $s \in \mathbb{N}$ there holds

$$\left(\frac{\partial}{\partial t} + V_s \cdot \nabla_{X_s} \right) f_\infty^{(s)}(t, Z_s) = \ell^{-1} C_{s+1}^0 f_\infty^{(s+1)}(t, Z_s) \tag{158}$$

in the sense of distributions.

Proof. Recall the free evolution $(T_s^0(t) f_\infty^{(s)})(Z_s) = f_\infty^{(s)}(X_s - V_s t, V_s)$, where $Z_s \in \mathbb{R}^{2ds}$. Subject to the estimates stated in the theorem, and the continuity of $f_\infty^{(s)}(t, Z_s)$, the weak form of the Boltzmann hierarchy is equivalent

to the following mild form:

$$f_{\infty}^{(s)}(t) = T_s^0(t) f_{\infty}^{(s)}(0) + \ell^{-1} \int_0^t T_s^0(t-\tau) C_{s+1}^0 f_{\infty}^{(s+1)}(\tau) d\tau \quad (159)$$

At this point it is convenient to change the coordinates. Let us define $G_{\infty}(t) = \left\{ g_{\infty}^{(s)}(t) \right\}_{s \geq 1}$ by $g_{\infty}^{(s)}(t) = T_s^0(-t) f_{\infty}^{(s)}(t)$, and write

$$V_{s+1}^0(\tau) = T_s^0(-\tau) C_{s+1}^0 T_{s+1}^0(\tau) \quad (160)$$

Then we have

$$g_{\infty}^{(s)}(t) = g_{\infty}^{(s)}(0) + \ell^{-1} \int_0^t V_{s+1}^0(\tau) g_{\infty}^{(s+1)}(\tau) d\tau \quad (161)$$

We record an explicit formula for the action of the operator $V_{s,s+1}^0(\tau)$:

$$V_{s+1}^0(\tau) = V_{s+1}^{0,+}(\tau) - V_{s+1}^{0,-}(\tau) \quad (162)$$

$$\begin{aligned} \left(V_{s+1}^{0,+}(\tau) g_{\infty}^{(s+1)}(t) \right) (Z_s) &= \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} [\omega \cdot (v_{s+1} - v_i)]_+ \times \\ &\times g_{\infty}^{(s+1)}(t, x_1, v_1, \dots, x_i - (v_i^* - v_i) \tau, v_i^*, \dots, \\ &\dots, x_s, v_s, x_i - (v_{s+1}^* - v_i) \tau, v_{s+1}^*) \end{aligned} \quad (163)$$

$$\begin{aligned} \left(V_{s+1}^{0,-}(\tau) g_{\infty}^{(s+1)}(t) \right) (Z_s) &= \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} [\omega \cdot (v_{s+1} - v_i)]_- \times \\ &\times g_{\infty}^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i, \dots, x_s, v_s, x_i - (v_{s+1} - v_i) \tau, v_{s+1}) \end{aligned} \quad (164)$$

We will prove pointwise bounds for the operators $V_{s+1}^{0,\pm}(\tau)$. If $0 < \beta' < \beta$, $\mu' < \mu$, $t, \tau \geq 0$, then we have:

$$\begin{aligned}
& \left| \left(e^{\mu' s} e^{\beta' (E_s(Z_s) + I_s(Z_s))} V_{s+1}^{0,+}(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right| \leq \\
& \leq \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta-\beta')E_s(Z_s)} e^{-(\mu-\mu')s} \times \\
& \times e^{-\frac{1}{2}\beta|v_{s+1}|^2} e^{\frac{1}{2}\beta(|x_i|^2 - |x_i - (v_i^* - v_i)\tau|^2 - |x_i - (v_{s+1}^* - v_i)\tau|^2)} e^{-\mu} \times \\
& \times e^{\mu(s+1)} e^{\frac{1}{2}\beta \sum_{i=1}^{s+1} |v_i|^2} e^{\frac{1}{2}\beta(|x_1|^2 + \dots + |x_i - (v_i^* - v_i)\tau|^2 + \dots + |x_s|^2 + |x_i - (v_{s+1}^* - v_i)\tau|^2)} \times \\
& \times \left| g_\infty^{(s+1)}(t, x_1, v_1, \dots, x_i - (v_i^* - v_i)\tau, v_i^*, \dots \right. \\
& \quad \left. \dots, x_s, v_s, x_i - (v_{s+1}^* - v_i)\tau, v_{s+1}^*) \right| \\
& \leq \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta-\beta')E_s(Z_s)} e^{-(\mu-\mu')s} \times \\
& \times e^{-\frac{1}{2}\beta|v_{s+1}|^2} e^{\frac{1}{2}\beta(|x_i|^2 - |x_i - (v_i^* - v_i)\tau|^2 - |x_i - (v_{s+1}^* - v_i)\tau|^2)} e^{-\mu} \times \\
& \times \left\| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z'_{s+1}) + I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t, Z'_{s+1}) \right\|_{L_{Z'_{s+1}}^\infty}
\end{aligned}$$

and similarly

$$\begin{aligned}
& \left| \left(e^{\mu' s} e^{\beta' (E_s(Z_s) + I_s(Z_s))} V_{s+1}^{0,-}(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right| \leq \\
& \leq \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta-\beta')E_s(Z_s)} e^{-(\mu-\mu')s} \times \\
& \times e^{-\frac{1}{2}\beta|v_{s+1}|^2} e^{-\frac{1}{2}\beta|x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
& \times e^{\mu(s+1)} e^{\frac{1}{2}\beta \sum_{i=1}^{s+1} |v_i|^2} e^{\frac{1}{2}\beta(|x_1|^2 + \dots + |x_i|^2 + \dots + |x_s|^2 + |x_i - (v_{s+1} - v_i)\tau|^2)} \times \\
& \times \left| g_\infty^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i, \dots, x_s, v_s, x_i - (v_{s+1} - v_i)\tau, v_{s+1}) \right| \\
& \leq \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta-\beta')E_s(Z_s)} e^{-(\mu-\mu')s} \times \\
& \times e^{-\frac{1}{2}\beta|v_{s+1}|^2} e^{-\frac{1}{2}\beta|x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
& \times \left\| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z'_{s+1}) + I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t, Z'_{s+1}) \right\|_{L_{Z'_{s+1}}^\infty}
\end{aligned}$$

The following identity follows from elementary manipulation:

$$|x_i|^2 + |x_i - (v_{s+1} - v_i)\tau|^2 - |x_i - (v_i^* - v_i)\tau|^2 - |x_i - (v_{s+1}^* - v_i)\tau|^2 = 0 \quad (165)$$

Therefore we obtain a bound on the full operator $V_{s+1}^0(\tau)$,

$$\begin{aligned}
& \left| \left(e^{\mu' s} e^{\beta'(E_s(Z_s) + I_s(Z_s))} V_{s+1}^0(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right| \leq \\
& \leq 2 \sum_{i=1}^s \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_{s+1} |v_{s+1} - v_i| e^{-(\beta - \beta')E_s(Z_s)} e^{-(\mu - \mu')s} \times \\
& \quad \times e^{-\frac{1}{2}\beta|v_{s+1}|^2} e^{-\frac{1}{2}\beta|x_i - (v_{s+1} - v_i)\tau|^2} e^{-\mu} \times \\
& \quad \times \left\| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z'_{s+1}) + I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t, Z'_{s+1}) \right\|_{L_{Z'_{s+1}}^\infty}
\end{aligned} \tag{166}$$

We use the following dispersive inequality [3]:

$$\|\zeta(x - vt, v)\|_{L_x^\infty L_v^1} \leq |t|^{-d} \|\zeta(x, v)\|_{L_x^1 L_v^\infty} \tag{167}$$

which implies the *pointwise* bound

$$\begin{aligned}
& \left| \left(e^{\mu' s} e^{\beta'(E_s(Z_s) + I_s(Z_s))} V_{s+1}^0(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right| \leq \\
& \leq C_d e^{-\mu} \beta^{-\frac{d}{2}} (1 + \tau)^{-d} \left(s^{\frac{1}{2}} E_s(Z_s)^{\frac{1}{2}} + s \beta^{-\frac{1}{2}} \right) e^{-(\beta - \beta')E_s(Z_s)} e^{-(\mu - \mu')s} \times \\
& \quad \times \left\| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z'_{s+1}) + I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t, Z'_{s+1}) \right\|_{L_{Z'_{s+1}}^\infty}
\end{aligned} \tag{168}$$

and therefore also implies

$$\begin{aligned}
& \left\| \left(e^{\mu' s} e^{\beta'(E_s(Z_s) + I_s(Z_s))} V_{s+1}^0(\tau) g_\infty^{(s+1)}(t) \right) (Z_s) \right\|_{L_{Z_s}^\infty} \leq \\
& \leq C_d e^{-\mu} \beta^{-\frac{d}{2}} (1 + \tau)^{-d} \left(\frac{1}{\sqrt{\beta - \beta'} \cdot \sqrt{\mu - \mu'}} + \frac{\beta^{-\frac{1}{2}}}{\mu - \mu'} \right) \times \\
& \quad \times \left\| e^{\mu(s+1)} e^{\beta(E_{s+1}(Z'_{s+1}) + I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t, Z'_{s+1}) \right\|_{L_{Z'_{s+1}}^\infty}
\end{aligned} \tag{169}$$

Fix a sequence of positive numbers r_0, r_1, r_2, \dots such that $0 < r_{k+1} < r_k$ and $\sum_{k=0}^\infty r_k = 1$. We define continuous decreasing functions $\beta(t), \mu(t)$, for $t \geq 0$:

$$\beta(t) = \beta_0 \cdot \left[1 - \frac{1}{2} \sum_{0 \leq k < n} r_k - \frac{1}{2} r_n (t - n) \right] \quad \forall \quad t \in [n, n+1) \tag{170}$$

$$\mu(t) = \mu_0 - \sum_{0 \leq k < n} r_k - r_n (t - n) \quad \forall \quad t \in [n, n+1) \tag{171}$$

Using the *pointwise* bound (166), we obtain

$$\begin{aligned}
& \left| e^{\mu(t)s} e^{\beta(t)(E_s(Z_s)+I_s(Z_s))} \ell^{-1} \int_0^t \left(V_{s+1}^0(\tau) g_\infty^{(s+1)}(\tau) \right) (Z_s) d\tau \right| \leq \\
& \leq C_d \ell^{-1} e^{-(\mu_0-1)} \left(\frac{\beta_0}{2} \right)^{-\frac{d}{2}} \left(s^{\frac{1}{2}} E_s(Z_s)^{\frac{1}{2}} + s \left(\frac{\beta_0}{2} \right)^{-\frac{1}{2}} \right) \times \\
& \quad \times \int_0^t (1+\tau)^{-d} e^{-(\beta(\tau)-\beta(t))E_s(Z_s)} e^{-(\mu(\tau)-\mu(t))s} d\tau \times \\
& \quad \times \left\| e^{\mu(t')(s+1)} e^{\beta(t')(E_{s+1}(Z'_{s+1})+I_{s+1}(Z'_{s+1}))} g_\infty^{(s+1)}(t', Z'_{s+1}) \right\|_{L_{t'}^\infty L_{Z'_{s+1}}^\infty} \\
& \hspace{15em} (172)
\end{aligned}$$

Then by a straightforward computation we have

$$\int_0^t (1+\tau)^{-d} e^{-(\beta(\tau)-\beta(t))E_s(Z_s)} e^{-(\mu(\tau)-\mu(t))s} d\tau \leq \frac{\sum_{k=0}^\infty r_k^{-1} (1+k)^{-d}}{s + \frac{\beta_0}{2} E_s(Z_s)} \quad (173)$$

Observe that if $d \geq 3$ then we may choose r_k such that $r_k \sim k^{-d+\frac{3}{2}}$ as $k \rightarrow \infty$, and $\sum_{k=0}^\infty r_k = 1$; then, we will also have $\sum_{k=0}^\infty r_k^{-1} (1+k)^{-d} < \infty$. Hence for $d \geq 3$ there holds

$$\begin{aligned}
& \left\| e^{\mu(t)s} e^{\beta(t)(E_s(Z_s)+I_s(Z_s))} \ell^{-1} \int_0^t \left(V_{s+1}^0(\tau) g_\infty^{(s+1)}(\tau) \right) (Z_s) d\tau \right\|_{L_t^\infty L_{Z_s}^\infty} \leq \\
& \leq C'_d \ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}} \times \\
& \quad \times \left\| e^{\mu(t)(s+1)} e^{\beta(t)(E_{s+1}(Z_{s+1})+I_{s+1}(Z_{s+1}))} g_\infty^{(s+1)}(t, Z_{s+1}) \right\|_{L_t^\infty L_{Z_{s+1}}^\infty} \\
& \hspace{15em} (174)
\end{aligned}$$

The Boltzmann hierarchy can be written in the following vector form:

$$G_\infty(t) = G_\infty(0) + \ell^{-1} \int_0^t V^0(\tau) G_\infty(\tau) d\tau \quad (175)$$

where $V^0(\tau)G_\infty(t) = \left\{ V_{s+1}^0(\tau) g_\infty^{(s+1)}(t) \right\}_{s \in \mathbb{N}}$. We work in the Banach space $(\mathcal{X}, \|\cdot\|)$ of sequences $G_\infty(t) = \left\{ g_\infty^{(s)}(t) \right\}_{s \in \mathbb{N}}$ with each function $g_\infty^{(s)}(t) : [0, \infty) \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ continuous and symmetric, and with norm

$$\|G_\infty\| = \sup_{t \geq 0} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} e^{\mu(t)s} e^{\beta(t)(E_s(Z_s)+I_s(Z_s))} \left| g_\infty^{(s)}(t, Z_s) \right| \quad (176)$$

Then we may define the operator $\mathcal{V} : \mathcal{X} \rightarrow \mathcal{X}$,

$$(\mathcal{V}G_\infty)(t) = \ell^{-1} \int_0^t V^0(\tau) G_\infty(\tau) d\tau \quad (177)$$

We may view the data $G_\infty(0)$ as an element of \mathcal{X} which simply does not depend on time. Then the Boltzmann hierarchy may be written as

$$G_\infty = G_\infty(0) + \mathcal{V}G_\infty \quad (178)$$

Since $\|\mathcal{V}\|_{\text{op}} \leq C'_d \ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$, as soon as $\ell^{-1} e^{-\mu_0} \beta_0^{-\frac{d+1}{2}}$ is sufficiently small we can invert this equation to give

$$G_\infty = (\mathcal{I} - \mathcal{V})^{-1} G_\infty(0) = \sum_{j=0}^{\infty} \mathcal{V}^j G_\infty(0) \quad (179)$$

which is the unique solution of the Boltzmann hierarchy. \square \square

Remark. We cannot apply the above argument, as written, in the case $d = 2$; this is due to the failure of integrability at large times. However, this is a technical restriction since Theorem 6.1 gives us *a priori* bounds for the BBGKY hierarchy, independent of N , for all $d \geq 2$. Indeed, a slightly different argument from the one above actually implies that Theorem 10.1 holds when $d = 2$ (see [21]); note that the only difference in their proof was that while they could not show that $\sum_j \|\mathcal{V}\|_{\text{op}}^j < \infty$, they could at least prove that $\sum_j \|\mathcal{V}^j G_\infty(0)\| < \infty$, under the same assumptions. Alternatively, for chaotic data, we can use the solvability of the Boltzmann equation near vacuum (see [11]), combined with the local well-posedness of the Boltzmann hierarchy; this line of reasoning would still be completely sufficient to reach the conclusions of Theorem 12.1 in the case $d = 2$.

To conclude this section, we quote a couple of local-in-time well-posedness results for the Boltzmann hierarchy. The proofs are well-known and similar to the proof presented above.

Theorem 10.2. *Suppose $F_\infty(0) = \{f_\infty^{(s)}(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f_\infty^{(s)}(0) : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ is continuous and symmetric, and for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$,*

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(0, Z_s) \right| e^{\beta_0[E_s(Z_s) + I_s(Z_s)]} e^{\mu_0 s} \leq 1 \quad (180)$$

Then there is a constant $C_d > 0$, depending only on d , such that if $T_L < C_d \ell e^{\mu_0} \beta_0^{\frac{d+1}{2}}$, then there exists a unique sequence $F_\infty(t) = \{f_\infty^{(s)}(t)\}_{s \in \mathbb{N}}$, with each $f_\infty^{(s)}(t, Z_s) : [0, T_L] \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ continuous and symmetric, such that

$$\sup_{0 \leq t \leq T_L} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(t, Z_s) \right| e^{\frac{1}{2}\beta_0[E_s(Z_s) + I_s((X_s - V_s t, V_s))]} e^{(\mu_0 - 1)s} \leq 2 \quad (181)$$

and for each $s \in \mathbb{N}$ there holds

$$\left(\frac{\partial}{\partial t} + V_s \cdot \nabla_{X_s} \right) f_\infty^{(s)}(t, Z_s) = \ell^{-1} C_{s+1}^0 f_\infty^{(s+1)}(t, Z_s) \quad (182)$$

in the sense of distributions, for $0 \leq t \leq T_L$.

Theorem 10.3. *Suppose $F_\infty(0) = \{f_\infty^{(s)}(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f_\infty^{(s)}(0) : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ is continuous and symmetric, and for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$,*

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1 \quad (183)$$

Then there is a constant $C_d > 0$, depending only on d , such that if $T_L < C_d \ell e^{\mu_0} \beta_0^{\frac{d+1}{2}}$, then there exists a unique sequence $F_\infty(t) = \{f_\infty^{(s)}(t)\}_{s \in \mathbb{N}}$, with each $f_\infty^{(s)}(t, Z_s) : [0, T_L] \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ continuous and symmetric, such that

$$\sup_{0 \leq t \leq T_L} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s)} e^{(\mu_0 - 1)s} \leq 2 \quad (184)$$

and for each $s \in \mathbb{N}$ there holds

$$\left(\frac{\partial}{\partial t} + V_s \cdot \nabla_{X_s} \right) f_\infty^{(s)}(t, Z_s) = \ell^{-1} C_{s+1}^0 f_\infty^{(s+1)}(t, Z_s) \quad (185)$$

in the sense of distributions, for $0 \leq t \leq T_L$.

11. CONSTRUCTION OF THE INITIAL DATA

We introduce the N -particle density f_N

$$f_N(0, Z_N) = \mathcal{Z}_N^{-1} \mathbf{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) \quad (186)$$

where \mathcal{Z}_N is the partition function,

$$\mathcal{Z}_N = \int_{\mathbb{R}^{2dN}} \mathbf{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) dZ_N \quad (187)$$

We also use the notation \mathcal{Z}_s for $1 \leq s \leq N$ (note carefully the implicit dependence on ε),

$$\mathcal{Z}_s = \int_{\mathbb{R}^{2ds}} \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) dZ_s \quad (188)$$

The proofs in this section are almost identical to those in the literature; we include them for the sake of completeness. [15]

Lemma 11.1. *For $1 \leq s < N$, and any probability density $f_0(x, v)$ on \mathbb{R}^{2d} with $f_0 \in L_x^\infty L_v^1$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ there holds*

$$\mathcal{Z}_{s+1} \geq \mathcal{Z}_s \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \right) \quad (189)$$

where B_1^d is the unit ball in \mathbb{R}^d and \mathcal{Z}_s is given by (188).

Proof. For $1 \leq s < N$, we have

$$\begin{aligned} \mathcal{Z}_{s+1} &= \int_{\mathbb{R}^{2d(s+1)}} \mathbf{1}_{Z_{s+1} \in \mathcal{D}_{s+1}} f_0^{\otimes(s+1)}(Z_{s+1}) dZ_{s+1} \\ &= \int_{\mathbb{R}^{2d(s+1)}} \mathbf{1}_{Z_s \in \mathcal{D}_s} \left(\prod_{i=1}^s \mathbf{1}_{|x_i - x_{s+1}| > \varepsilon} \right) f_0^{\otimes(s+1)}(Z_{s+1}) dZ_{s+1} \\ &= \int_{\mathbb{R}^{2ds}} \mathbf{1}_{Z_s \in \mathcal{D}_s} \left[\int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left(\prod_{i=1}^s \mathbf{1}_{|x_i - x_{s+1}| > \varepsilon} \right) dz_{s+1} \right] f_0^{\otimes s}(Z_s) dZ_s \end{aligned}$$

We bound the quantity in brackets from below, uniformly in Z_s .

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left(\prod_{i=1}^s \mathbf{1}_{|x_i - x_{s+1}| > \varepsilon} \right) dz_{s+1} \\ &\geq \int_{\mathbb{R}^{2d}} f_0(z_{s+1}) \left(1 - \sum_{i=1}^s \mathbf{1}_{|x_i - x_{s+1}| \leq \varepsilon} \right) dz_{s+1} \\ &\geq 1 - s\varepsilon^d |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \\ &\geq 1 - N\varepsilon^{d-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \\ &= 1 - \ell^{-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \end{aligned}$$

We have used the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ in the last step. Finally we are able to conclude, for $1 \leq s < N$,

$$\mathcal{Z}_{s+1} \geq \mathcal{Z}_s \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \right) \quad (190)$$

as claimed. \square \square

Lemma 11.2. For $1 \leq s < N$, and any probability density $f_0(x, v)$ on \mathbb{R}^{2d} with $f_0 \in L_x^\infty L_v^1$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ there holds

$$1 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \leq \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \right)^{-s} \quad (191)$$

where B_1^d is the unit ball in \mathbb{R}^d and \mathcal{Z}_s is given by (188).

Proof. For the first inequality, we note that clearly $\mathcal{Z}_N \leq \mathcal{Z}_s \mathcal{Z}_{N-s}$, then use the fact that $\mathcal{Z}_s \leq 1$. The second inequality follows directly from Lemma 11.1 by induction on s . \square \square

Lemma 11.3. For $1 \leq s \leq N$, and any probability density $f_0(x, v)$ on \mathbb{R}^{2d} with $f_0 \in L_x^\infty L_v^1$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ there holds

$$f_N^{(s)}(0, Z_s) \leq \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \left(1 - \ell^{-1} |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon \right)^{-s} \quad (192)$$

where B_1^d is the unit ball in \mathbb{R}^d and $f_N^{(s)}(0)$ is the marginal of the data $f_N(0)$ given by (186).

Proof. We proceed by computation.

$$\begin{aligned}
f_N^{(s)}(0, Z_s) &= \int_{\mathbb{R}^{2d(N-s)}} \mathcal{Z}_N^{-1} \mathbf{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(0, Z_N) dZ_{(s+1):N} \\
&\leq \int_{\mathbb{R}^{2d(N-s)}} \mathcal{Z}_N^{-1} \mathbf{1}_{Z_s \in \mathcal{D}_s} \mathbf{1}_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} f_0^{\otimes N}(0, Z_N) dZ_{(s+1):N} \\
&= \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s)
\end{aligned}$$

Then the result follows from Lemma 11.2. \square \square

Lemma 11.4. For $1 \leq s \leq N$, and any probability density $f_0(x, v)$ on \mathbb{R}^{2d} with $f_0 \in L_x^\infty L_v^1$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ there holds

$$f_N^{(s)}(0, Z_s) \geq \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \left(1 - (s+1)\ell^{-1}|B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon\right) \quad (193)$$

where B_1^d is the unit ball in \mathbb{R}^d and $f_N^{(s)}(0)$ is the marginal of the data $f_N(0)$ given by (186).

Proof. We proceed by computation.

$$\begin{aligned}
f_N^{(s)}(0, Z_s) &= \int_{\mathbb{R}^{2d(N-s)}} \mathcal{Z}_N^{-1} \mathbf{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N) dZ_{(s+1):N} \\
&= \int_{\mathbb{R}^{2d(N-s)}} \mathcal{Z}_N^{-1} \mathbf{1}_{Z_s \in \mathcal{D}_s} \mathbf{1}_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} \times \\
&\quad \times \left(\prod_{1 \leq i \leq s} \prod_{s < j \leq N} \mathbf{1}_{|x_i - x_j| > \varepsilon} \right) f_0^{\otimes N}(Z_N) dZ_{(s+1):N} \\
&= \mathcal{Z}_N^{-1} \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \int_{\mathbb{R}^{2d(N-s)}} \mathbf{1}_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} \times \\
&\quad \times \left(\prod_{1 \leq i \leq s} \prod_{s < j \leq N} \mathbf{1}_{|x_i - x_j| > \varepsilon} \right) f_0^{\otimes(N-s)}(Z_{(s+1):N}) dZ_{(s+1):N}
\end{aligned}$$

Now observe that

$$\prod_{1 \leq i \leq s} \prod_{s < j \leq N} \mathbf{1}_{|x_i - x_j| > \varepsilon} \geq 1 - \sum_{1 \leq i \leq s} \sum_{s < j \leq N} \mathbf{1}_{|x_i - x_j| \leq \varepsilon} \quad (194)$$

Then again, for $1 \leq i \leq s$, $s < j \leq N$, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2d(N-s)}} \mathbf{1}_{Z_{(s+1):N} \in \mathcal{D}_{N-s}} \mathbf{1}_{|x_i - x_j| \leq \varepsilon} f_0^{\otimes(N-s)}(Z_{(s+1):N}) dZ_{(s+1):N} &\leq \\
&\leq \mathcal{Z}_{N-s-1} \varepsilon^d |B_1^d| \|f_0\|_{L_x^\infty L_v^1}
\end{aligned} \quad (195)$$

Therefore,

$$\begin{aligned}
f_N^{(s)}(0, Z_s) &\geq \mathcal{Z}_N^{-1} \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \times \\
&\quad \times \left[\mathcal{Z}_{N-s} - s(N-s) \mathcal{Z}_{N-s-1} \varepsilon^d |B_1^d| \|f_0\|_{L_x^\infty L_v^1} \right]
\end{aligned} \quad (196)$$

We use Lemma 11.1, Lemma 11.2, and the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ to conclude

$$f_N^{(s)}(0, Z_s) \geq \mathbf{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \left(1 - (s+1)\ell^{-1}|B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon\right) \quad (197)$$

□

□

Corollary 11.5. *For any probability density $f_0(x, v) > 0$ on \mathbb{R}^{2d} with $f_0 \in L_x^\infty L_v^1$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, if N is sufficiently large, then simultaneously for all $1 \leq s \leq N$ there holds*

$$\left\| \mathbf{1}_{Z_s \in \mathcal{D}_s} \left(\frac{f_N^{(s)}(0, Z_s)}{f_0^{\otimes s}(Z_s)} - 1 \right) \right\|_{L_{Z_s}^\infty} \leq \left[\left(1 - \ell^{-1}|B_1^d| \|f_0\|_{L_x^\infty L_v^1} \varepsilon\right)^{-(s+1)} - 1 \right] \quad (198)$$

where $f_N^{(s)}(0)$ is the marginal of the data $f_N(0)$ given by (186).

Corollary 11.6. *Let f_0 be a probability density on \mathbb{R}^{2d} with*

$$\left\| f_0(x, v) e^\mu e^{\frac{1}{2}\beta|v|^2} \right\|_{L_{x,v}^\infty} \leq 1 \quad (199)$$

for some $\beta > 0, \mu \in \mathbb{R}$. Then for any $\mu' < \mu$ we have for all sufficiently large N in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$ the estimate

$$\sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta E_s(Z_s)} e^{\mu' s} \leq 1 \quad (200)$$

where $f_N^{(s)}(0)$ is the marginal of the data $f_N(0)$ given by (186).

12. LOCAL-IN-TIME CONVERGENCE PROOF

The main result of this section is a local-in-time propagation of chaos result for the BBGKY hierarchy. We will use the stability result from Section 8 in order to prove uniform convergence on a set of “good” phase points.

Theorem 12.1. *Suppose $F_N(t) = \{f_N^{(s)}(t)\}_{1 \leq s \leq N}$ is a solution of the BBGKY hierarchy (33), subject to the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$, and with each function $f_N^{(s)} : [0, \infty) \times \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ symmetric under particle interchange. Further suppose $F_\infty(0) = \{f_\infty^{(s)}(0)\}_{s \in \mathbb{N}}$ is a sequence of functions such that each $f_\infty^{(s)}(0) : \mathbb{R}^{2ds} \rightarrow \mathbb{R}$ is continuous and symmetric. Assume that for some $\beta_0 > 0, \mu_0 \in \mathbb{R}$,*

$$\sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1 \quad (201)$$

$$\sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1 \quad (202)$$

Then there is a constant $C_d > 0$, depending only on d , such that if $T_L < C_d l e^{\mu_0} \beta_0^{\frac{d+1}{2}}$, then all of the following are true:

(i) $F_N(t)$ satisfies the bound

$$\sup_{0 \leq t \leq T_L} \sup_{1 \leq s \leq N} \sup_{Z_s \in \mathcal{D}_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s)} e^{(\mu_0 - 1)s} \leq 1 \quad (203)$$

(ii) the Boltzmann hierarchy has a unique continuous symmetric solution $F_\infty(t)$, $t \in [0, T_L]$, satisfying the bound

$$\sup_{0 \leq t \leq T_L} \sup_{s \in \mathbb{N}} \sup_{Z_s \in \mathbb{R}^{2ds}} \left| f_\infty^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s)} e^{(\mu_0 - 1)s} \leq 2 \quad (204)$$

(iii) if $f_\infty^{(s)}(0) = f_0^{\otimes s} \forall s \in \mathbb{N}$ for some Lipschitz-continuous probability density $f_0(x, v)$, and likewise $\left\{ \left\{ f_N^{(s)}(0) \right\}_{1 \leq s \leq N} \right\}_{N \in \mathbb{N}}$ is nonuniformly f_0 -chaotic (see Section 2), then $f_\infty^{(s)}(t) = f_t^{\otimes s} \forall s \in \mathbb{N}$ for $t \in [0, T_L]$ where f_t solves Boltzmann's equation, and $\left\{ \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N} \right\}_{N \in \mathbb{N}}$ is nonuniformly f_t -chaotic for $t \in [0, T_L]$.

Proof. The local well-posedness of the Boltzmann hierarchy, and the bounds (203-204), are direct consequences of Theorem 5.1 and Theorem 10.3.

We introduce a smooth cut-off function $\chi : [0, \infty) \rightarrow \mathbb{R}$, decreasing, with $0 \leq \chi \leq 1$, $\chi(z) = 1$ for $0 \leq z \leq 1$, $\|\chi'\|_\infty \leq 2$, and $\chi(z) = 0$ for $z \geq 2$. Given parameters $R > 0$ and $n \in \mathbb{N}$, we define

$$f_{N,n,R}^{(s)}(0, Z_s) = f_N^{(s)}(0, Z_s) \mathbf{1}_{1 \leq s \leq n} \chi \left(\frac{1}{R^2} E_s(Z_s) \right) \quad (205)$$

and let $F_{N,n,R}(0) = \left\{ f_{N,n,R}^{(s)}(0) \right\}_{1 \leq s \leq N}$. We let $F_{N,n,R}(t)$ be the solution of the BBGKY hierarchy (33) with initial data $F_{N,n,R}(0)$. Similarly, given initial data $F_\infty(0) = \left\{ f_\infty^{(s)}(0) \right\}_{s \in \mathbb{N}}$, define

$$f_{\infty,n,R}^{(s)}(0, Z_s) = f_\infty^{(s)}(0, Z_s) \mathbf{1}_{1 \leq s \leq n} \chi \left(\frac{1}{R^2} E_s(Z_s) \right) \quad (206)$$

and let $F_{\infty,n,R}(0) = \left\{ f_{\infty,n,R}^{(s)}(0) \right\}_{s \in \mathbb{N}}$. We let $F_{\infty,n,R}(t)$ be the solution of the Boltzmann hierarchy with data $F_{\infty,n,R}(0)$. Using Theorem 5.1 and Theorem 10.3, and the linearity of the BBGKY and Boltzmann hierarchies, and dividing C_d by $e \cdot 2^{\frac{d+1}{2}}$ in the statement of the theorem, we immediately obtain the following estimates:

$$\sup_{\substack{1 \leq s \leq N \\ t \in [0, T_L] \\ Z_s \in \mathcal{D}_s}} \left| \left(f_N^{(s)} - f_{N,n,R}^{(s)} \right) (t, Z_s) \right| e^{\frac{1}{4} \beta_0 E_s(Z_s)} e^{(\mu_0 - 2)s} \leq e^{-\frac{1}{2} \beta_0 R^2} + e^{-n} \quad (207)$$

$$\sup_{\substack{s \in \mathbb{N} \\ t \in [0, T_L] \\ Z_s \in \mathbb{R}^{2ds}}} \left| \left(f_{\infty}^{(s)} - f_{\infty, n, R}^{(s)} \right) (t, Z_s) \right| e^{\frac{1}{4}\beta_0 E_s(Z_s)} e^{(\mu_0 - 2)s} \leq 2 \left(e^{-\frac{1}{2}\beta_0 R^2} + e^{-n} \right) \quad (208)$$

The remainder of the proof consists of comparing the two functions $f_{N, n, R}^{(s)}(t)$ and $f_{\infty, n, R}^{(s)}(t)$.

We have the following Duhamel series:

$$\begin{aligned} f_{N, n, R}^{(s)}(t, Z_s) &= \sum_{k=0}^{n-s} a_{N, k, s} \times \\ &\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \quad (209) \\ &\times \left(b_{s, s+k}[\cdot] f_{N, n, R}^{(s+k)}(0, Z_{s, s+k}[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned}$$

$$\begin{aligned} f_{\infty, n, R}^{(s)}(t, Z_s) &= \sum_{k=0}^{n-s} \ell^{-k} \times \\ &\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \quad (210) \\ &\times \left(b_{s, s+k}^0[\cdot] f_{\infty, n, R}^{(s+k)}(0, Z_{s, s+k}^0[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned}$$

where

$$a_{N, k, s} = \frac{(N-s)!}{(N-s-k)!} \varepsilon^{k(d-1)} \quad (211)$$

It is not hard to show that all terms appearing in the *finite* series (209-210) are finite for all $t \geq 0$. Note that the expression (209) is meaningful as a measurable function if the data is integrable and compactly supported (see [21] for a detailed proof of this fact), whereas the expression (210) makes sense due to the continuity of the data $F_{\infty, n, R}(0)$.

Let us now define a new function, $\tilde{f}_{N, n, R}^{(s)}(t)$, which is closely related to $f_{N, n, R}^{(s)}(t)$.

$$\begin{aligned} \tilde{f}_{N, n, R}^{(s)}(t, Z_s) &= \sum_{k=0}^{n-s} \ell^{-k} \times \\ &\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \quad (212) \\ &\times \left(b_{s, s+k}[\cdot] f_{N, n, R}^{(s+k)}(0, Z_{s, s+k}[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned}$$

Note that $|a_{N,k,s} - \ell^{-k}| \leq [1 - (1 - \frac{n}{N})^n] \ell^{-k}$ for $0 \leq k \leq n - s$; therefore,

$$\begin{aligned} & \left| \tilde{f}_{N,n,R}^{(s)}(t, Z_s) - f_{N,n,R}^{(s)}(t, Z_s) \right| \leq \left[1 - \left(1 - \frac{n}{N} \right)^n \right] \times \\ & \sum_{k=0}^{n-s} \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \ell^{-k} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \\ & \times \left(|b_{s,s+k}[\cdot]| \left| f_{N,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) \right| \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned} \quad (213)$$

To estimate the series in (213), we recall that $f_{N,n,R}^{(s+k)}(0)$ is absolutely bounded by $e^{-\mu_0(s+k)}$ and is supported in the set $E_{s+k}(Z_{s+k}) \leq 2R^2$. Hence, due to energy conservation, all the iterated integrals appearing in (213) range over compact sets and we can evaluate the maximum possible contributions explicitly. Note that this is a significant over-estimate since we are not using the exponential decay of $f_{N,n,R}^{(s+k)}(0)$ at large energies; nevertheless, this crude estimate will suffice for the proof. We obtain

$$\begin{aligned} & \left| \tilde{f}_{N,n,R}^{(s)}(t, Z_s) - f_{N,n,R}^{(s)}(t, Z_s) \right| \leq \\ & \leq \left[1 - \left(1 - \frac{n}{N} \right)^n \right] e^{-\mu_0 s} \exp \left[C_d \ell^{-1} n R^{d+1} e^{-\mu_0 t} \right] \end{aligned} \quad (214)$$

Observe that the right-hand side of (214) tends to zero as $N \rightarrow \infty$ when n, R, Z_s, t are all held fixed.

Let us now fix $Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^\eta$, $t \in [0, T_L]$, with $E_s(Z_s) \leq 2R^2$. Let us pick parameters $\eta, \theta, \alpha, y > 0$ such that $R > \eta$ and $\sin \theta > c_d y^{-1} \varepsilon$, where c_d is as in the statement of Proposition 8.3. Let us define

$$\mathcal{A}_{n,R} = \sum_{k=0}^n C_d^k \ell^{-k} R^{k(d+1)} n^k e^{-\mu_0 k} T_L^k \quad (215)$$

where the constant C_d is to be chosen in the next step. Then, by repeated application of Proposition 8.3, we can construct sets $\{\mathcal{B}_k\}_{k=0}^{n-s}$, dependent on (Z_s, t) , with

$$\mathcal{B}_k \subset \left([0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{N} \right)^k \quad (216)$$

such that

$$\begin{aligned} & \sum_{k=0}^{n-s} \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \ell^{-k} \int_0^t \cdots \int_0^{t_{k-1}} \int_{(B_{2R}^d)^k} \int_{(\mathbb{S}^{d-1})^k} \mathbf{1}_{\mathcal{B}_k} \prod_{m=1}^k d\omega_m dv_{s+m} dt_m \times \\ & \times \left(|b_{s,s+k}[\cdot]| \left| f_{N,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) \right| \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \leq \\ & \leq e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \left[\alpha + \frac{y}{\eta T_L} + C_{d,\alpha} \left(\left(\frac{\eta}{R} \right)^{d-1} + \theta^{(d-1)/2} \right) \right] \end{aligned} \quad (217)$$

$$\begin{aligned}
& \sum_{k=0}^{n-s} \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \ell^{-k} \int_0^t \cdots \int_0^{t_{k-1}} \int_{(B_{2R}^d)^k} \int_{(\mathbb{S}^{d-1})^k} \mathbf{1}_{\mathcal{B}_k} \prod_{m=1}^k d\omega_m dv_{s+m} dt_m \times \\
& \times \left(|b_{s,s+k}^0[\cdot]| \left| f_{\infty,n,R}^{(s+k)}(0, Z_{s,s+k}^0[\cdot]) \right| \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \leq \\
& \leq e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \left[\alpha + \frac{y}{\eta T_L} + C_{d,\alpha} \left(\left(\frac{\eta}{R} \right)^{d-1} + \theta^{(d-1)/2} \right) \right]
\end{aligned} \tag{218}$$

and such that whenever

$$\begin{aligned}
& \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \\
& \in \left(\left([0, T_L] \times B_{2R}^d \times \mathbb{S}^{d-1} \times \mathbb{N} \right)^k \setminus \mathcal{B}_k \right) \cap \{0 \leq t_k \leq \cdots \leq t_1 \leq t\}
\end{aligned} \tag{219}$$

there holds

$$\left| \left(Z_{s,s+k}[\cdot] - Z_{s,s+k}^0[\cdot] \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \right|_{\infty} \leq k\varepsilon \tag{220}$$

$$b_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] = b_{s,s+k}^0 \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \tag{221}$$

$$Z_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \in \mathcal{K}_{s+k} \cap \mathcal{U}_{s+k}^{\eta} \tag{222}$$

Here $|Z_j|_{\infty} = \sup_{i=1,\dots,j} \max(|x_i|, |v_i|)$.

Remark. The sets \mathcal{B}_k collect all integration points for which the Duhamel series (210) and (212) fail to agree. At the remaining points, the pseudo-trajectories $Z_{s,s+k}[\dots]$ and $Z_{s,s+k}^0[\dots]$ are identical, up to $\mathcal{O}(\varepsilon)$ perturbations of the particles' spatial positions. These perturbations are harmless because the Boltzmann hierarchy propagates smoothness forwards in time.

As long as we are away from \mathcal{B}_k , we can use the triangle inequality:

$$\begin{aligned}
& \left| \left(f_{\infty,n,R}^{(s+k)}(0, Z_{s,s+k}^0[\cdot]) - f_{N,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_j \right] \right| \\
& \leq \left| \left(f_{\infty,n,R}^{(s+k)}(0, Z_{s,s+k}^0[\cdot]) - f_{\infty,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_j \right] \right| \\
& + \left| \left(f_{\infty,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) - f_{N,n,R}^{(s+k)}(0, Z_{s,s+k}[\cdot]) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_j \right] \right|
\end{aligned} \tag{223}$$

We can easily control the first term using the regularity assumption on $f_{\infty}^{(j)}(0)$ combined with the stability estimate (220). On the other hand, due to (222), in order to control the second term, we only need to estimate $\left| f_{\infty}^{(s+k)} - f_N^{(s+k)} \right|$ on $\mathcal{K}_{s+k} \cap \mathcal{U}_{s+k}^{\eta}$.

Remark. Carefully observe that it is entirely possible that $Z_{s,s+k}^0[\dots] \notin \mathcal{K}_{s+k} \cap \mathcal{U}_{s+k}^{\eta}$, even away from \mathcal{B}_k . This is because in the construction of \mathcal{B}_k , we never ruled out events wherein two particles only “barely” miss each other under the backwards flow.

Now we easily obtain

$$\begin{aligned}
& \sup_{\substack{0 \leq t \leq T_L \\ Z_s \in \mathbb{R}^{2ds}}} \left| \left(f_N^{(s)} - f_\infty^{(s)} \right) (t, Z_s) \right| \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^\eta} \mathbf{1}_{E_s(Z_s) \leq 2R^2} \\
& \leq 3e^{-(\mu_0-2)s} \left(e^{-\frac{1}{2}\beta_0 R^2} + e^{-n} \right) + \\
& + \left[1 - \left(1 - \frac{n}{N} \right)^n \right] e^{-\mu_0 s} e^{C_d \ell^{-1} n R^{d+1}} e^{-\mu_0 T_L} + \\
& + 2e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \left[\alpha + \frac{y}{\eta T_L} + C_{d,\alpha} \left(\frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right] + \\
& + C_d n^{\frac{5}{2}} R^{-1} e^{|\mu_0|n} \varepsilon e^{C_d \ell^{-1} n R^{d+1}} e^{-\mu_0 T_L} + \\
& + C_d n^2 \varepsilon e^{C_d \ell^{-1} n R^{d+1}} e^{-\mu_0 T_L} \sup_{\substack{1 \leq j \leq n \\ Z_j \in \mathbb{R}^{2dj}}} \left| \nabla_{Z_j} f_\infty^{(j)}(0, Z_j) \right|_2 \mathbf{1}_{E_j(Z_j) \leq 2R^2} + \\
& + C_d e^{C_d \ell^{-1} n R^{d+1}} e^{-\mu_0 T_L} \sup_{\substack{1 \leq j \leq n \\ Z_j \in \mathbb{R}^{2dj}}} \left| \left(f_N^{(j)} - f_\infty^{(j)} \right) (0, Z_j) \right| \mathbf{1}_{Z_j \in \mathcal{K}_j \cap \mathcal{U}_j^\eta} \mathbf{1}_{E_j(Z_j) \leq 2R^2}
\end{aligned} \tag{224}$$

where $|\nabla_{Z_s} f^{(s)}|_2^2 = \sum_{i=1}^s \left(|\nabla_{x_i} f^{(s)}|^2 + |\nabla_{v_i} f^{(s)}|^2 \right)$. According to the definition of nonuniform f_0 -chaoticity, we may let $\eta = \varepsilon^\kappa$ for some fixed $\kappa \in (0, 1)$. We will then let $y = \varepsilon^{(1+\kappa)/2}$ and $\theta \sim \varepsilon^{(1-\kappa)/4}$; in particular, the constraint $\sin \theta \geq c_d y^{-1} \varepsilon$ is satisfied. Now let $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$ simultaneously in the Boltzmann-Grad scaling, $N \varepsilon^{d-1} = \ell^{-1}$, and use the fact that $f_\infty^{(j)}(0) = f_0^{\otimes j}$ and that $\left\{ \left\{ f_N^{(j)}(0) \right\}_{1 \leq j \leq N} \right\}_{N \in \mathbb{N}}$ is nonuniformly f_0 -chaotic.

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \sup_{\substack{0 \leq t \leq T_L \\ Z_s \in \mathbb{R}^{2ds}}} \left| \left(f_N^{(s)} - f_\infty^{(s)} \right) (t, Z_s) \right| \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq 2R^2} \\
& \leq 3e^{-(\mu_0-2)s} \left(e^{-\frac{1}{2}\beta_0 R^2} + e^{-n} \right) + 2e^{-\mu_0 s} n^2 \mathcal{A}_{n,R} \alpha
\end{aligned} \tag{225}$$

Since $\alpha > 0$ is arbitrary we have

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} \sup_{\substack{0 \leq t \leq T_L \\ Z_s \in \mathbb{R}^{2ds}}} \left| \left(f_N^{(s)} - f_\infty^{(s)} \right) (t, Z_s) \right| \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq 2R^2} \\
& \leq 3e^{-(\mu_0-2)s} \left(e^{-\frac{1}{2}\beta_0 R^2} + e^{-n} \right)
\end{aligned} \tag{226}$$

Since n is arbitrary, the second term on the right-hand side can be thrown away. On the other hand, the left-hand side only increases as R increases, so we can throw away the first term on the right-hand side as well. Since the Boltzmann hierarchy propagates chaoticity, we have $f_\infty^{(s)}(t) = f_t^{\otimes s}$ for

$t \in [0, T_L]$; hence,

$$\limsup_{N \rightarrow \infty} \sup_{\substack{0 \leq t \leq T_L \\ Z_s \in \mathbb{R}^{2ds}}} \left| f_N^{(s)}(t, Z_s) - f_t^{\otimes s}(Z_s) \right| \mathbf{1}_{Z_s \in \mathcal{K}_s \cap \mathcal{U}_s^{\eta(\varepsilon)}} \mathbf{1}_{E_s(Z_s) \leq 2R^2} = 0 \quad (227)$$

We conclude that $\left\{ \left\{ f_N^{(s)}(t) \right\}_{1 \leq s \leq N} \right\}_{N \in \mathbb{N}}$ is nonuniformly f_t -chaotic for $t \in [0, T_L]$. \square

Remark. We can deduce part (i) of Theorem 2.1 directly from Theorem 12.1 by splitting the time interval $[0, T]$ into smaller intervals $[0, T_L]$, $[T_L, 2T_L]$, etc., for some sufficiently small time T_L .

APPENDIX A. PARTIAL PROOF OF PART (ii) OF THEOREM 2.1

The proof consists of three parts. The first part is the introduction of an unsymmetric Boltzmann-Enskog hierarchy; we show that this auxiliary hierarchy propagates *partial* factorization. The second part is to show that a certain class of pseudo-trajectories for the BBGKY dynamics coincide (with high probability) with the corresponding pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy. The third part is to add up all the sources of error pointwise, as in Section 12. We outline the proof of the first step, provide full technical estimates for the *pre-collisional* case of the second step (the post-collisional case is similar), and skip the third step (which is tedious yet straightforward). We remark that a much more general version of the same result (accounting for correlations of any finite number of particles) is currently under investigation. This result and the proof were largely inspired by the techniques of M. Pulvirenti and S. Simonella. [33]

A.1. An Unsymmetric Boltzmann-Enskog Hierarchy. We are going to construct an infinite hierarchy of equations which tracks correlations between the first $m-1$ labeled particles while ignoring *all* correlations between the remaining particles. Clearly, such a hierarchy cannot preserve symmetry between all particles. Nevertheless, we will be able to prove a partial factorization property which will be the key to part (ii) of Theorem 2.1. The factorization property we will prove for the resulting hierarchy is that if $s \geq m \geq 2$ then

$$g_\varepsilon^{(s)}(t) = g_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t)^{\otimes (s-m+1)} \quad (228)$$

if such factorization holds at the initial time; here $g_\varepsilon(t)$ is the solution to a Boltzmann-Enskog type equation.

Let us introduce the unsymmetric s -particle phase space, where $m \geq 2$ is fixed and $s \geq m-1$:

$$\tilde{\mathcal{D}}_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{ds} \times \mathbb{R}^{ds} \mid \forall 1 \leq i < j \leq m-1, |x_i - x_j| > \varepsilon \right\} \quad (229)$$

Observe that in the definition of $\tilde{\mathcal{D}}_s$, we only enforce an exclusion condition between the first $m-1$ particles. We do *not* have exclusion for any pair

of particles for which *at least one* particle index is greater than $m - 1$. We define the collision operators,

$$\tilde{C}_{s+1} = \sum_{i=1}^s \left(\tilde{C}_{i,s+1}^+ - \tilde{C}_{i,s+1}^- \right) \quad (230)$$

where

$$\begin{aligned} \tilde{C}_{i,s+1}^+ g_\varepsilon^{(s+1)}(t, Z_s) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\omega \cdot (v_{s+1} - v_i)]_+ \times \\ &\quad \times g_\varepsilon^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}^*) d\omega dv_{s+1} \end{aligned} \quad (231)$$

$$\begin{aligned} \tilde{C}_{i,s+1}^- g_\varepsilon^{(s+1)}(t, Z_s) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\omega \cdot (v_{s+1} - v_i)]_- \times \\ &\quad \times g_\varepsilon^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i, \dots, x_s, v_s, x_i + \varepsilon \omega, v_{s+1}) d\omega dv_{s+1} \end{aligned} \quad (232)$$

and

$$\begin{aligned} v_i^* &= v_i + \omega \omega \cdot (v_{s+1} - v_i) \\ v_{s+1}^* &= v_{s+1} - \omega \omega \cdot (v_{s+1} - v_i) \end{aligned} \quad (233)$$

The function $g_\varepsilon^{(s)}(t, Z_s)$ is defined for $0 \leq t < T$ and $Z_s \in \tilde{\mathcal{D}}_s$, $s \geq m - 1$, as the solution to the following hierarchy of equations:

$$(\partial_t + V_s \cdot \nabla_{X_s}) g_\varepsilon^{(s)}(t) = \ell^{-1} \tilde{C}_{s+1} g_\varepsilon^{(s+1)}(t) \quad (\text{if } s \geq m - 1) \quad (234)$$

with boundary condition

$$g_\varepsilon^{(s)}(t, Z_s^*) = g_\varepsilon^{(s)}(t, Z_s) \quad \text{a.e. } (t, Z_s) \in [0, T) \times \partial \tilde{\mathcal{D}}_s \quad (235)$$

and initial conditions $g_\varepsilon^{(s)}(0, Z_s)$ defined for $s \geq m - 1$ and $Z_s \in \tilde{\mathcal{D}}_s$. We also introduce the function $g_\varepsilon(t, x, v)$ ($t \geq 0$, $x, v \in \mathbb{R}^d$) which is defined to be the solution to the equation

$$(\partial_t + v \cdot \nabla_x) g_\varepsilon(t) = \ell^{-1} \tilde{C}_2(g_\varepsilon(t) \otimes g_\varepsilon(t)) \quad (236)$$

with prescribed initial data $g_\varepsilon(0)$.

We now introduce a mild form for (234-235). For any $s \geq m - 1$, let $\tilde{T}_s(t)$ denote the strongly continuous semigroup on $L^2(\tilde{\mathcal{D}}_s)$ with generator $-V_s \cdot \nabla_{X_s}$ and specular reflection boundary conditions along $\partial \tilde{\mathcal{D}}_s$. The operators $\tilde{T}_s(t)$ extend to other functional spaces by standard density arguments. Then, under sufficiently strong regularity conditions, the hierarchy (234-235) is equivalent to the following hierarchy written in mild form:

$$g_\varepsilon^{(s)}(t) = \tilde{T}_s(t) g_\varepsilon^{(s)}(0) + \ell^{-1} \int_0^t \tilde{T}_s(t - \tau) \tilde{C}_{s+1} g_\varepsilon^{(s+1)}(\tau) d\tau \quad (s \geq m - 1) \quad (237)$$

Following Lanford's fixed point argument, we are able to prove existence and uniqueness of solutions to (237) on a short time interval. However, under the conditions of Lanford's proof, the distributional form (234-235) and the mild form (237) are equivalent, so we are free to work with either

formulation for our computations. Note that, in a similar fashion, we can define mild solutions for (236), and solutions can be constructed on a short time interval by a fixed point argument.

We will state a well-posedness theorem for (234-235) so that we can refer to the result later. The proof follows Lanford's fixed point argument so we omit it.

Proposition A.1. *Fix an integer $m \geq 2$. Let $\{g_\varepsilon^{(s)}(0)\}_{s \geq m-1}$ be a sequence of functions, with each $g_\varepsilon^{(s)}(0)$ defined for $Z_s \in \tilde{\mathcal{D}}_s$. Furthermore, suppose that there exists $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$ such that*

$$\sup_{s \geq m-1} \sup_{Z_s \in \tilde{\mathcal{D}}_s} e^{\mu_0 s} e^{\beta_0 E_s(Z_s)} \left| g_\varepsilon^{(s)}(0, Z_s) \right| \leq 1 \quad (238)$$

Then there exists a constant $C_d > 0$ such that the following is true: If $T_L < C_d e^{\mu_0} \beta_0^{\frac{d+1}{2}}$ then there exists a unique sequence of functions $\{g_\varepsilon^{(s)}(t)\}_{s \geq m-1}$ defined for $t \in [0, T_L]$ such that (i), (ii), and (iii) below all hold.

(i) For any bounded open set $\mathcal{O} \subset [0, T_L] \times \tilde{\mathcal{D}}_s$, we have $(\partial_t + V_s \cdot \nabla_{X_s}) g_\varepsilon^{(s)} \in L^1(\mathcal{O})$.

(ii) We have the bound:

$$\sup_{s \geq m-1} \sup_{t \in [0, T_L]} \sup_{Z_s \in \tilde{\mathcal{D}}_s} e^{(\mu_0 - 1)s} e^{\frac{1}{2}\beta_0 E_s(Z_s)} \left| g_\varepsilon^{(s)}(t, Z_s) \right| \leq 2 \quad (239)$$

(iii) The sequence $\{g_\varepsilon^{(s)}(t)\}_{s \geq m-1}$ solves (234-235) in the sense of distributions on $[0, T_L]$ with initial data $\{g_\varepsilon^{(s)}(0)\}_{s \geq m-1}$; note that the equation is well-defined thanks to (i) and (ii).

We now turn to the main result of this section:

Proposition A.2. *Fix an integer $m \geq 2$. Let $\{g_\varepsilon^{(s)}(t)\}_{s \geq m-1}$ be a sequence of functions, with each $g_\varepsilon^{(s)}(t, Z_s)$ defined for $(t, Z_s) \in [0, T) \times \tilde{\mathcal{D}}_s$. Let $g_\varepsilon(t, x, v)$ be defined for $t \in [0, T)$ and $x, v \in \mathbb{R}^d$. Further suppose that there exists $\beta_T > 0$ and $\mu_T \in \mathbb{R}$ such that*

$$\sup_{s \geq m-1} \sup_{t \in [0, T)} \sup_{Z_s \in \tilde{\mathcal{D}}_s} e^{\mu_T s} e^{\beta_T E_s(Z_s)} \left| g_\varepsilon^{(s)}(t, Z_s) \right| \leq 1 \quad (240)$$

and

$$\sup_{t \in [0, T)} \sup_{x, v \in \mathbb{R}^d} e^{\mu_T t} e^{\frac{1}{2}\beta_T |v|^2} |g_\varepsilon(t, x, v)| \leq 1 \quad (241)$$

and that $(\partial_t + V_s \cdot \nabla_{X_s}) g_\varepsilon^{(s)} \in L^1(\mathcal{O})$ for any bounded open set $\mathcal{O} \subset [0, T) \times \tilde{\mathcal{D}}_s$. Then, if $\{g_\varepsilon^{(s)}(t)\}_{s \geq m-1}$ solve (234-235), $g_\varepsilon(t)$ solves (236), and

$$g_\varepsilon^{(s)}(0) = g_\varepsilon^{(m-1)}(0) \otimes g_\varepsilon(0)^{\otimes (s-m+1)} \quad (242)$$

for all $s \geq m$, then for $t \in [0, T)$ there holds

$$g_\varepsilon^{(s)}(t) = g_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t)^{\otimes(s-m+1)} \quad (243)$$

for all $s \geq m$.

Proof. We proceed by constructing a solution of the unsymmetric Boltzmann-Enskog hierarchy (234-235) with the desired property; then, the conclusion follows by uniqueness. Let $T_L < C_d \ell e^{\mu T} \beta_T^{\frac{d+1}{2}}$, where C_d is the constant appearing in Proposition A.1.

Recall that $g_\varepsilon(t)$ is the solution to (236), with initial data $g_\varepsilon(0)$. Let us now define $u_\varepsilon^{(m-1)}(t)$ to be the solution of the following equation, for $0 \leq t \leq T_L$:

$$(\partial_t + V_{m-1} \cdot \nabla_{X_{m-1}}) u_\varepsilon^{(m-1)}(t) = \ell^{-1} \tilde{C}_m \left(u_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t) \right) \quad (244)$$

with boundary condition $u_\varepsilon^{(m-1)}(t, Z_{m-1}^*) = u_\varepsilon^{(m-1)}(t, Z_{m-1})$ along $[0, T_L] \times \partial \tilde{\mathcal{D}}_{m-1}$, and initial data $g_\varepsilon^{(m-1)}(0)$. The existence and uniqueness for (244) on a time interval of size T_L follows from a modified version of Lanford's fixed point argument; moreover, the solution obeys the following bound:

$$\sup_{t \in [0, T_L]} \sup_{Z_{m-1} \in \tilde{\mathcal{D}}_{m-1}} e^{2(\mu T - 1)} e^{\frac{1}{2} \beta_T E_{m-1}(Z_{m-1})} \left| u_\varepsilon^{(m-1)}(t, Z_{m-1}) \right| \leq 2 \quad (245)$$

Having defined $u_\varepsilon^{(m-1)}(t)$, let us define, for $s \geq m$,

$$u_\varepsilon^{(s)}(t) = u_\varepsilon^{(m-1)}(t) \otimes g_\varepsilon(t)^{\otimes(s-m+1)} \quad (246)$$

Now it is straightforward to verify that the sequence $\left\{ u_\varepsilon^{(s)}(t) \right\}_{s \geq m-1}$ satisfies (234-235) for $t \in [0, T_L]$; by uniqueness, we conclude that $g_\varepsilon^{(s)}(t) = u_\varepsilon^{(s)}(t)$ for all $s \geq m-1$ and $t \in [0, T_L]$.

We can iterate the same argument on the time intervals $[T_L, 2T_L]$, $[2T_L, 3T_L]$, etc., until we have covered the full time interval $[0, T)$. \square \square

A.2. Series Solution for the Unsymmetric Boltzmann-Enskog Hierarchy. We will develop a series expansion and corresponding pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy (234-235). The main differences between the unsymmetric Boltzmann-Enskog hierarchy and the BBGKY hierarchy are twofold: first, the former is an infinite hierarchy, whereas the latter is finite; and second, the former tracks correlations between $m-1$ particles, whereas the latter tracks correlations between *all* particles. Since the two hierarchies are so similar, the developments in this section will be almost identical to those of Section 7. Nevertheless, there are a few subtle differences which are important in our proof, so in the interest of completeness we repeat the construction in this case.

The main point we wish to emphasize is that there is a new dynamics, given by a measurable measure-preserving map $\tilde{\psi}_s^t : \tilde{\mathcal{D}}_s \rightarrow \tilde{\mathcal{D}}_s$, with the

property that

$$\left(\tilde{T}_s(t)g^{(s)}\right)(Z_s) = g^{(s)}\left(\tilde{\psi}_s^{-t}Z_s\right) \quad (247)$$

where $g^{(s)}(Z_s)$ is an arbitrary measurable function with finite integral, and \tilde{T}_s is the transport operator appearing in the mild form (237) of the unsymmetric Boltzmann-Enskog hierarchy. The dynamics $\tilde{\psi}_s^t$ forces collisions between the first $m-1$ labeled particles, whereas any pair of particles with (i, j) with $1 \leq i \leq s$ and $m \leq j \leq s$ may pass through each other without colliding. We will need to use $\tilde{\psi}_s^t$ in place of ψ_s^t in the construction of pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy.

Similar to the BBGKY hierarchy, we can write down an iterated Duhamel series for the unsymmetric Boltzmann-Enskog hierarchy (237), like so:

$$\begin{aligned} g_\varepsilon^{(s)}(t) &= \sum_{k=0}^{\infty} \ell^{-k} \times \\ &\times \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \tilde{T}_s(t-t_1) \tilde{C}_{s+1} \dots \tilde{T}_{s+k}(t_k) g_\varepsilon^{(s+k)}(0) dt_k \dots dt_1 \\ &\quad (\text{if } s \geq m-1) \end{aligned} \quad (248)$$

Notice that the collision operators \tilde{C}_{s+1} have replaced the collision operators C_{s+1} which appear in the Duhamel series for the BBGKY hierarchy, and the transport operators \tilde{T}_s have replaced T_s . The collision operators \tilde{C}_{s+1} *do not* enforce any exclusion condition, as can be seen from (230-232); this fact will have to be reflected in the construction of pseudo-trajectories.

Fix an integer $m \geq 2$ and let $s \geq m-1$. We will be defining the symbols

$$\tilde{Z}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \quad (249)$$

where $Z_s \in \overline{\mathcal{D}}_s$, $0 \leq t_k < \dots < t_2 < t_1 < t$, $i_1 \in \{1, 2, \dots, s\}$, $i_2 \in \{1, 2, \dots, s+1\}$, \dots , $i_k \in \{1, 2, \dots, s+k-1\}$, $v_{s+j} \in \mathbb{R}^d$, and $\omega_j \in \mathbb{S}^{d-1}$.

Given $Z_s \in \overline{\mathcal{D}}_s$ and $t > 0$ we define

$$\tilde{Z}_{s,s} [Z_s, t] = \tilde{\psi}_s^{-t} Z_s \quad (250)$$

and $\tilde{Z}_{s,s} [Z_s, 0] = Z_s$. If the symbol

$$\tilde{Z}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \in \overline{\mathcal{D}}_{s+k} \quad (251)$$

is defined, then for all $\tau > 0$ we define

$$\begin{aligned} \tilde{Z}_{s,s+k} \left[Z_s, t + \tau; \{t_j + \tau, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] &= \\ &= \tilde{\psi}_{s+k}^{-\tau} \tilde{Z}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+k}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned} \quad (252)$$

Now suppose that the symbol

$$\tilde{Z}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}) \in \tilde{\mathcal{D}}_{s+k} \quad (253)$$

is defined, $t_{k+1} = 0$, $v_{s+k+1} \in \mathbb{R}^d$, $\omega_{k+1} \in \mathbb{S}^{d-1}$, and $i_{k+1} \in \{1, 2, \dots, s+k\}$. Further suppose that $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \leq 0$. Then we define

$$\tilde{Z}_{s,s+k+1} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^{k+1} \right] = \left(X'_{s+k}, V'_{s+k}, x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1} \right) \quad (254)$$

Similarly, suppose that the symbol

$$\tilde{Z}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] = (X'_{s+k}, V'_{s+k}) \in \tilde{\mathcal{D}}_{s+k} \quad (255)$$

is defined, $t_{k+1} = 0$, $v_{s+k+1} \in \mathbb{R}^d$, $\omega_{k+1} \in \mathbb{S}^{d-1}$, and $i_{k+1} \in \{1, 2, \dots, s+k\}$. Further suppose that $\omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) > 0$. Then we define

$$\begin{aligned} \tilde{Z}_{s,s+k+1} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^{k+1} \right] = \\ = \left(x'_1, v'_1, \dots, x'_{i_{k+1}}, v'_{i_{k+1}} + \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}), \dots, x'_s, v'_s, \right. \\ \left. x'_{i_{k+1}} + \varepsilon \omega_{k+1}, v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \right) \end{aligned} \quad (256)$$

Now we define the iterated collision kernel, again using induction. If $Z_s \in \tilde{\mathcal{D}}_s$ and $t \geq 0$ we define

$$\tilde{b}_{s,s} [Z_s, t] = 1 \quad (257)$$

If the symbol

$$\tilde{b}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \quad (258)$$

is defined and $\tau > 0$ then we define

$$\begin{aligned} \tilde{b}_{s,s+k} \left[Z_s, t + \tau; \{t_j + \tau, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] = \\ = \tilde{b}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned} \quad (259)$$

If the symbol

$$\tilde{b}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \quad (260)$$

is defined, and $t_{k+1} = 0$, $v_{s+k+1} \in \mathbb{R}^d$, $\omega_{k+1} \in \mathbb{S}^{d-1}$, and $i_{k+1} \in \{1, 2, \dots, s+k\}$ then we define

$$\begin{aligned} \tilde{b}_{s,s+k+1} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^{k+1} \right] = \\ = \tilde{b}_{s,s+k} \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \times \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}) \end{aligned} \quad (261)$$

We now have the following identity which holds pointwise for $Z_s \in \overline{\mathcal{D}}_s$ and $t \geq 0$:

$$\begin{aligned} g_\varepsilon^{(s)}(t, Z_s) &= \sum_{k=0}^{\infty} \ell^{-k} \times \\ &\times \sum_{i_1=1}^s \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{dk}} \int_{(\mathbb{S}^{d-1})^k} \left(\prod_{m=1}^k d\omega_m dv_{s+m} dt_m \right) \times \\ &\times \left(\tilde{b}_{s,s+k}[\cdot] g_\varepsilon^{(s+k)} \left(0, \tilde{Z}_{s,s+k}[\cdot] \right) \right) \left[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k \right] \end{aligned} \quad (262)$$

A.3. Stability of pseudo-trajectories. This subsection is concerned purely with the psuedo-trajectories generated by the BBGKY hierarchy. We are going to show that, if Z_s is such that all but the first two particles have free trajectories under the backwards particle flow¹¹, then adding a particle preserves this property with high probability. In fact we will prove the *pre-collisional* part of this estimate; the post-collisional part is dealt with similarly. This result is important because it allows us to compare pseudo-trajectories for the BBGKY hierarchy with those of the unsymmetric Boltzmann-Enskog hierarchy. Then it is straightforward to conclude that partial factorization is propagated by the BBGKY hierarchy, because by Proposition A.2, partial factorization is propagated by the unsymmetric Boltzmann-Enskog hierarchy.

Remark. We fix $m = 3$ in order to justify the result $f_N^{(s)}(t) \approx f_N^{(2)}(t) \otimes f_t^{\otimes(s-2)}$ on the set $\mathcal{G}_s \cap \hat{\mathcal{U}}_s^{\eta(\varepsilon)}$, introduced in Section 2, under the assumption that the entire sequence $\{F_N(0)\}_N$ is 2-nonuniformly f_0 -chaotic.

We will require the following sets:

$$\mathcal{G}_s = \left\{ Z_s = (X_s, V_s) \in \overline{\mathcal{D}}_s \left| \begin{array}{l} \forall \tau > 0, \forall 3 \leq i \leq s, \\ (\psi_s^{-\tau} Z_s)_i = (x_i - v_i \tau, v_i) \\ \text{and, } \forall \tau > 0, \forall 1 \leq i \leq 2, \forall 3 \leq j \leq s, \\ |(x_i - x_j) - (v_i - v_j) \tau| > \varepsilon \end{array} \right. \right\} \quad (263)$$

$$\mathcal{V}_s^\eta = \left\{ (Z_s, Z'_s) \in \overline{\mathcal{D}}_s \times \overline{\mathcal{D}}_s \left| \begin{array}{l} \inf_{1 \leq i \neq j \leq s} |v_i - v'_j| > \eta \\ \text{and} \\ \inf_{1 \leq i \leq s : |v_i - v'_i| \neq 0} |v_i - v'_i| > \eta \end{array} \right. \right\} \quad (264)$$

$$\hat{\mathcal{U}}_s^\eta = \left\{ Z_s = (X_s, V_s) \in \mathcal{U}_s^\eta \left| \forall \tau, \tau' > 0, (\psi_s^{-\tau} Z_s, \psi_s^{-\tau'} Z_s) \in \mathcal{V}_s^\eta \right. \right\} \quad (265)$$

Note carefully that $Z_s \in \mathcal{G}_s$ *does not* guarantee that the backwards trajectory $\{\psi_s^{-t} Z_s\}_{t \geq 0}$ is free.

¹¹including the possibilities that the first two particles collide *or* “pass through” each other, or miss entirely

We are ready to state the main result for this section.

Proposition A.3. *There is a constant $c_d > 0$ such that all the following holds: Assume that*

$$\begin{aligned} Z_{s,s+k} [Z_s, t; t_1, \dots, t_k; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] = \\ = (X'_{s+k}, V'_{s+k}) \in \mathcal{G}_{s+k} \cap \hat{\mathcal{U}}_{s+k}^\eta \end{aligned} \quad (266)$$

and $E_{s+k}(Z'_{s+k}) \leq 2R^2$ with $\eta < R$; then,

(i) for all $\tau \geq 0$ we have

$$\begin{aligned} Z_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau; v_{s+1}, \dots, v_{s+k}; \omega_1, \dots, \omega_k; i_1, \dots, i_k] \\ \in \mathcal{G}_{s+k} \cap \hat{\mathcal{U}}_{s+k}^\eta \end{aligned} \quad (267)$$

(ii) for any $i_{k+1} \in \{1, 2, \dots, s+k\}$, and for any $\alpha, y > 0$ and $\theta \in (0, \frac{\pi}{2})$ such that $\sin \theta > c_d y^{-1} \varepsilon$, there exists a measurable set $\mathcal{B} \subset [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, which may depend on Z_s, t , and $\{t_j, v_{s+j}, \omega_j, i_j\}_{j=1}^k$, such that

$$\begin{aligned} \forall T > 0, \\ \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_{d,s,k} T R^d \left[\alpha + \frac{y}{\eta T} + C_{d,\alpha} \left(\frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right] \end{aligned} \quad (268)$$

and

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] \\ \in \mathcal{G}_{s+k+1} \cap \hat{\mathcal{U}}_{s+k+1}^\eta \end{aligned} \quad (269)$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in ([0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1}) \setminus \mathcal{B}$.

Proof. Claim (i) is trivial so we turn to claim (ii). The first step is to delete particle addition times for which particles are too concentrated in space. However, it will not be enough to delete times for which particles at a *single* time-slice are nearby. Instead, in order to eventually control *all* possible recollisions arising from the collisional dynamics, we gather together *all* line segments generated by the flow prior to time τ and project them via free flight to land on the τ time-slice. (See Figure 1.) Then we ask that the entire set of phase points generated in this way does not concentrate in space. For $t' \in \mathbb{R}$ and $Z_s^0 = (X_s^0, V_s^0) \in \mathbb{R}^{ds} \times \mathbb{R}^{ds}$ we define

$$\hat{\psi}_s^{t'} Z_s^0 = (X_s^0 + V_s^0 t', V_s^0) \quad (270)$$

Also, we define $Z'_{s+k}(\tau; t') = \hat{\psi}_{s+k}^{t'} \left(\psi_{s+k}^{-(\tau+t')} Z'_{s+k} \right)$. If $Z_s^0, Z_s^1 \in \mathbb{R}^{2ds}$ then we define

$$d_X(Z_s^0, Z_s^1) = \min \left(\inf_{1 \leq i \neq j \leq s} |x_i^0 - x_j^1|, \inf_{1 \leq i \leq s : |x_i^0 - x_i^1| \neq 0} |x_i^0 - x_i^1| \right) \quad (271)$$

$$\mathcal{B}_I = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that } \tau = 0 \text{ or } \right. \\ \left. \exists t', t'' \geq 0 : d_X(Z'_{s+k}(\tau; t'), Z'_{s+k}(\tau; t'')) \leq y \right\} \quad (272)$$

We can easily estimate the measure of \mathcal{B}_I due to the condition $Z'_{s+k} \in \mathcal{G}_{s+k}$; it suffices to consider (at most) two possible line segments for each of the first two particles (corresponding to whether the particles are allowed to collide, or pass through each other, or miss entirely), and for each $3 \leq i \leq s+k$, the *unique* backwards line segment available to one of the i th particle. (See Figure 1.) Distinct line segments are compared pairwise to find collisions or near-collisions. Since $Z'_{s+k} \in \hat{\mathcal{U}}_{s+k}^\eta$, any two line segments can only be within a distance y (along a fixed time slice τ) for a time $\Delta\tau$ of order $y\eta^{-1}$ (this is where we explicitly use the integral in the creation time τ). We have

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_I} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_{d,s,k} R^d \eta^{-1} y \quad (273)$$

At this point it is useful to distinguish between pre-collisional and post-collisional configurations for the added particle. Therefore we introduce two sets,

$$\mathcal{A}^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that } \right. \\ \left. \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau; 0)) > 0 \right\} \quad (274)$$

$$\mathcal{A}^- = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that } \right. \\ \left. \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau; 0)) \leq 0 \right\} \quad (275)$$

We also delete collisions which are close to grazing:

$$\mathcal{B}_{II} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \text{ such that } \right. \\ \left. \left| \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau; 0)) \right| \leq (\sin \alpha) \left| v_{s+k+1} - v'_{i_{k+1}}(\tau; 0) \right| \right\} \quad (276)$$

We have

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{II}} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_d T R^d \alpha \quad (277)$$

The pre-collisional configurations and post-collisional configurations are dealt with separately.

Pre-collisional configurations. We must guarantee that the $(s+k+1)$ -particle state is in $\hat{\mathcal{U}}_{s+k+1}^\eta$ at the time of particle creation. Let us define

$$\mathcal{B}_{III}^- = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that } \right. \\ \left. \exists i \in \{1, 2, \dots, s+k\}, t' \geq 0 : |v_{s+k+1} - v'_i(\tau; t)| \leq \eta \right\} \quad (278)$$

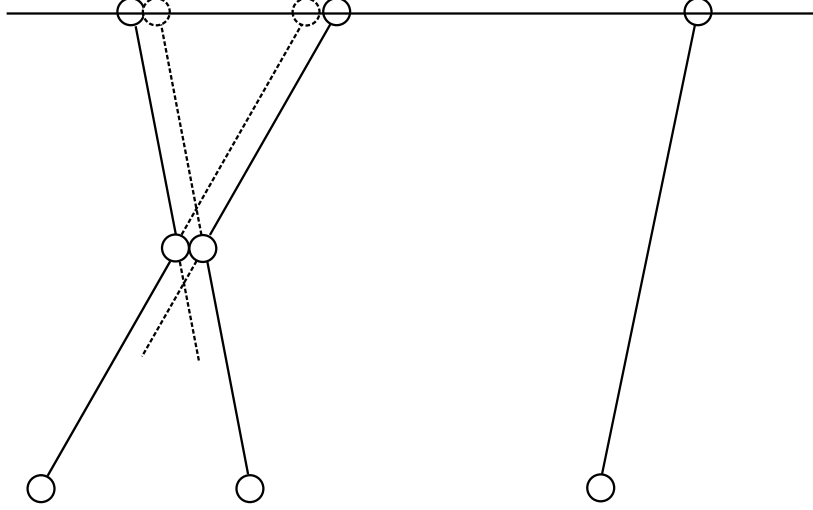


FIGURE 1. Particles are labelled increasing from left to right; the horizontal line is the τ time slice in the proof, and time increases from the bottom to the top of the diagram. Particles 1 and 2 may collide, or may pass through each other. Also, if the two particles *do* collide, then we must trace their *collisionally transformed paths* back to the τ time slice (which here is in the *future* relative to the collision). Hence particles 1 and 2 have “ghost” particles which must be accounted for in the proof; the ghosts are just as dangerous as the real particles as far as recollisions are concerned, because they eventually coincide with the true pseudo-trajectories. Note that the diagram is *not typical* in the sense that the condition $Z'_{s+k} \in \hat{\mathcal{U}}_{s+k}^\eta$ would usually prevent the ghost of particle 1 from landing near particle 2 at the τ time-slice (for most values of τ). This is a very important point because if we adjoin a new particle, say “4”, to particle 2 at time τ then the literal situation in the diagram *is not* proven to have small probability for recollisions by our strategy! (In particular \mathcal{B}_I does not have small measure.) Hence our proof says nothing about configurations where two particles are correlated via a *head on* collision.

If $(\tau, v_{s+k+1}, \omega_{k+1}) \notin \mathcal{B}_{III}^-$, and the $(s+k+1)$ particle’s backwards trajectory is free (which follows from the next step), we can be sure that the $(s+k+1)$ -particle state is in $\hat{\mathcal{U}}_{s+k+1}^\eta$. We have

$$\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{III}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d,s,k} T \eta^d \quad (279)$$

Finally we need to make sure that the backwards trajectory of the added particle is free; this means that the $s + k + 1$ particle never collides with or overlaps any of the particles appearing in Figure 1 under the backwards particle flow. Let us define

$$\mathcal{B}_{IV}^- = \left\{ \begin{array}{l} (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \text{ such that} \\ \exists i \in \{1, 2, \dots, s+k\}, t' \geq 0 : \\ \frac{(x'_{i_{k+1}}(\tau; 0) + \varepsilon\omega - x'_i(\tau; t')) \cdot (v_{s+k+1} - v'_i(\tau; t'))}{|x'_{i_{k+1}}(\tau; 0) + \varepsilon\omega - x'_i(\tau; t')| |v_{s+k+1} - v'_i(\tau; t')|} \geq \cos \theta \end{array} \right\} \quad (280)$$

We have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{IV}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_{d,s,k} T R^d \theta^{d-1} \end{aligned} \quad (281)$$

To conclude, we let $\mathcal{B}^- = \mathcal{B}_1 \cup \mathcal{B}_{II} \cup \mathcal{B}_{III}^- \cup \mathcal{B}_{IV}^-$; then we have

$$\begin{aligned} \int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} \mathbf{1}_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\ \leq C_{d,s,k} T R^d \left[\alpha + \frac{y}{\eta T} + \left(\frac{\eta}{R} \right)^d + \theta^{d-1} \right] \end{aligned} \quad (282)$$

Then again, by assumption, $\sin \theta > c_d y^{-1} \varepsilon$; by choosing c_d sufficiently large we may guarantee that

$$\begin{aligned} Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \dots, t_k + \tau, 0; v_{s+1}, \dots, v_{s+k}, v_{s+k+1}; \\ \omega_1, \dots, \omega_k, \omega_{k+1}; i_1, \dots, i_k, i_{k+1}] \\ \in \mathcal{G}_{s+k+1} \cap \hat{\mathcal{U}}_{s+k+1}^\eta \end{aligned} \quad (283)$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus \mathcal{B}^-$.

Post-collisional configurations. *Sketch.* The only difference compared to the case of pre-collisional configurations is that we must account for the collisional change of variables. This is slightly subtle because it means (for instance) that if we adjoin the $s + k + 1$ particle to the 2nd particle in a post-collisional configuration then, possibly, the interaction between the first two particles will be destroyed. This is no problem for the proof because the set \mathcal{G}_{s+k} was chosen to guarantee that the destruction of the interaction between particles 1 and 2 *does not* create recollisions with the particles $3 \leq i \leq s + k$.¹² The estimates can be carried out very similarly to the proof of Proposition 8.3, accounting for a few extra contributions as in the pre-collisional case of Proposition A.3 above. \square \square

¹²This is exactly the point of the proof which is very hard to handle for correlations of three or more particles; the two particle case is trivial.

APPENDIX B. ILLNER'S ESTIMATE

There is a close connection between bounds on the hard sphere BBGKY hierarchy and collision statistics for the underlying N -particle system. As a rule, we should be extremely pessimistic about bounding the number of collisions for N identical hard spheres in a uniform way. We refer the reader to [8], where an upper bound like $\left(32N^{\frac{3}{2}}\right)^{N^2}$ is proven. On the other hand, Illner [20] has proven that certain *weighted sums* over collisions may grow only polynomially in N ; these dynamical bounds translate directly into bounds on the BBGKY hierarchy. We remark that a polynomial bound of the type

$$\sup_{0 \leq t \leq T} \left\| e^{\mu s} e^{\beta E_s} f_N^{(s)}(t, Z_s) \right\|_{L_{Z_s}^\infty} \lesssim N \quad (284)$$

was indispensable in the recent derivation of the Stokes-Fourier limit in two dimensions. [4] The goal of this appendix is to show that similar bounds may be available in the nonlinear regime; in particular, we will show that there is some weak control over the behavior of the marginals on certain very singular sets. We make no claim as to the originality of the results in this appendix; we merely wish to point out a particular direction of investigation regarding Lanford's theorem which has not been fully explored.

We will prove the following proposition; a different proof may be found in [12]. The proof in [12] shows very clearly that the proposition follows essentially from the ideas in [20].

Proposition B.1. *Consider the system of N identical hard spheres of diameter ε set in \mathbb{R}^d . For each $N \in \mathbb{N}$, let $f_N(0)$ be an initial probability density on \mathcal{D}_N , which we assume to be symmetric under particle interchange, and let $f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N)$. Let $f_N^{(s)}(t)$, $1 \leq s \leq N$, denote the s -marginal of $f_N(t)$. Further assume that $f_N(0)$ is smooth and compactly supported in the interior of \mathcal{D}_N . Let us define*

$$K_x^N = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{2} f_N^{(1)}(0, x, v) dx dv \quad (285)$$

$$K_v^N = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v|^2}{2} f_N^{(1)}(0, x, v) dx dv \quad (286)$$

Then for all $2 \leq s \leq N$, there holds

$$\begin{aligned} & \sum_{1 \leq i < j \leq s} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{ds} \times \mathbb{R}^{d(s-1)} \times \mathbb{S}^{d-1}} |\omega \cdot (v_j - v_i)|^2 \times \\ & \quad \times f_N^{(s)}(t, \dots, x_i, v_i, \dots, x_i + \varepsilon \omega, v_j, \dots) d\omega dX_s^{(j)} dV_s dt \leq \\ & \quad \leq \frac{C_d}{N \varepsilon^d} s(s-1) (K_x^N K_v^N)^{\frac{1}{2}} \end{aligned} \quad (287)$$

where C_d depends only on the dimension d .

Proof. We will prove (287) for $s = 2$ only; the general case then follows by the fact that the functions $\{f_N^{(s)}(t)\}_{1 \leq s \leq N}$ are the marginals of the symmetric probability density $f_N(t)$. Similarly we may consider only $t \geq 0$ by time-reversibility.

Due to the symmetry of f_N and conservation of energy, we easily deduce, for any $t \geq 0$:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_N^{(1)}(t, x, v) dx dv = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_N^{(1)}(0, x, v) dx dv \quad (288)$$

Additionally, using Lemma 3.2 with $s = N$, and the symmetry of f_N , we obtain for any $t \geq 0$:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - vt|^2 f_N^{(1)}(t, x, v) dx dv \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_N^{(1)}(0, x, v) dx dv \quad (289)$$

We observe the identity

$$\left(\frac{\partial}{\partial t} + V_N \cdot \nabla_{X_N} \right) [\mathcal{Y}_N(Z_N) - 2tE_N(Z_N)] = 0 \quad (290)$$

which holds for all $Z_N \in \mathcal{D}_N$. Multiply both sides of (290) by $f_N(t, Z_N)$ and integrate to obtain:

$$\int_0^T \int_{\mathcal{D}_N} \left\{ \left(\frac{\partial}{\partial t} + V_N \cdot \nabla_{X_N} \right) [\mathcal{Y}_N(Z_N) - 2tE_N(Z_N)] \right\} f_N(t, Z_N) dZ_N dt = 0 \quad (291)$$

Next we integrate by parts in time and space, and use the fact that $f_N(t, Z_N)$ satisfies Liouville's equation. Then we are left with only the boundary contributions along $[0, T] \times \partial\mathcal{D}_N$ and $\{0\} \times \mathcal{D}_N \cup \{T\} \times \mathcal{D}_N$. These boundary integrals can all be expressed in terms of the marginals $f_N^{(1)}(t)$ and $f_N^{(2)}(t)$ due to the symmetry of $f_N(t)$ and the fact that the boundary condition $f_N(t, Z_N^*) = f_N(t, Z_N)$ holds a.e. $(t, Z_N) \in [0, T] \times \partial\mathcal{D}_N$.

$$\begin{aligned} & N \int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot v - |v|^2 T) f_N^{(1)}(T, x, v) dx dv - N \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot v f_N^{(1)}(0, x, v) dx dv \\ &= C_d \frac{N(N-1)}{2} \varepsilon^d \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |\omega \cdot (v_2 - v_1)|^2 \times \\ & \quad \times f_N^{(2)}(t, x_1, v_1, x_1 + \varepsilon\omega, v_2) d\omega dx_1 dv_1 dv_2 dt \end{aligned} \quad (292)$$

Note that a factor of ε^{d-1} appears due to the surface measure on a ball of radius ε , whereas the last power of ε arises from the jump in the function $\mathcal{Y}_N(Z_N)$ across a collision. The jump in $\mathcal{Y}_N(Z_N)$ also accounts for the appearance of the *square* of the collision kernel.

We will now estimate the moments of $f_N^{(1)}(t)$ in (292). Indeed, for any $\lambda > 0$, we clearly have

$$\begin{aligned} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot v f_N^{(1)}(0, x, v) dx dv \right| &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x \cdot v| f_N^{(1)}(0, x, v) dx dv \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\lambda |x|^2}{2} + \frac{|v|^2}{2\lambda} \right) f_N^{(1)}(0, x, v) dx dv \\ &\leq \lambda^{-1} (\lambda^2 K_x^N + K_v^N) \end{aligned}$$

On the other hand, we may use (288, 289) to estimate:

$$\begin{aligned} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot v - |v|^2 T) f_N^{(1)}(T, x, v) dx dv \right| &= \\ &= \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (x - vT) \cdot v f_N^{(1)}(T, x, v) dx dv \right| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |(x - vT) \cdot v| f_N^{(1)}(T, x, v) dx dv \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\lambda |x - vT|^2}{2} + \frac{|v|^2}{2\lambda} \right) f_N^{(1)}(T, x, v) dx dv \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\lambda |x|^2}{2} + \frac{|v|^2}{2\lambda} \right) f_N^{(1)}(0, x, v) dx dv \\ &= \lambda^{-1} (\lambda^2 K_x^N + K_v^N) \end{aligned}$$

These bounds combined with (292) lead us to

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |\omega \cdot (v_2 - v_1)|^2 f_N^{(2)}(t, x_1, v_1, x_1 + \varepsilon \omega, v_2) d\omega dx_1 dv_1 dv_2 dt \\ \leq \frac{C_d}{N \varepsilon^d} \lambda^{-1} (\lambda^2 K_x^N + K_v^N) \end{aligned} \tag{293}$$

The conclusion follows by optimizing over $\lambda > 0$. \square \square

Remark. In the Boltzmann-Grad scaling $N \varepsilon^{d-1} = \ell^{-1}$ for fixed $\ell > 0$, the right hand side of (287) is $\mathcal{O}(\varepsilon^{-1})$, which is in turn no larger than $\mathcal{O}(N)$. Also note that the left hand side of (287) is very similar to the L_{t, Z_s}^1 space-time norm of the (gain-only) BBGKY collision operator; the crucial difference is that the quantity in (287) contains the *square* of the collision kernel. Hence, (287) says that if *all* collisions (at the N particle level) satisfy a bound like $\{|\omega \cdot (v_2 - v_1)| \geq c_N\}$ for some fixed $c_N > 0$, then the L_{t, Z_s}^1 norm of the collision operator is at most $\mathcal{O}(\varepsilon^{-1} c_N^{-1})$. This is a very crude localization but it illustrates conveniently the meaning of Proposition B.1; in any case, the c_N^{-1} loss can be avoided by using weak topologies, as in [9]. Note that if it was possible to eliminate the ε^{-1} loss, and simultaneously replace the quantity on the left by the L_{t, Z_s}^1 norm of the (gain-only) BBGKY collision operator, then by passing to the limit we would extract an extraordinary

new bound for Boltzmann's equation *in terms of conserved energies only*; unsurprisingly, this is impossible, as we show next.

B.1. Optimality. We ask whether Proposition B.1 is optimal for general densities $f_N(0)$, in the sense that the scaling $N^{-1}\varepsilon^{-d}$ cannot be improved. We will show that the answer is yes. Colloquially, our results imply that, in the Boltzmann-Grad limit, the BBGKY collision operator cannot be controlled in L^1 , *uniformly in N* , using just the conserved energies of the problem. This result may be interpreted as verification that some control *beyond* the level of energy (for example, entropy dissipation) is a *necessary ingredient* for any improvement of Lanford's theorem. The reason is that finite second moments in x and v do not by themselves prevent spatial concentration of particles; if the density becomes too concentrated, the approximation of binary collisions is simply not valid anymore. These are standard observations in the theory of the space-inhomogeneous Boltzmann equation, but the counter-examples provided here are illustrative nonetheless.

First, by obvious reductions, we may set $\varepsilon = 1$ and $s = 2$; then, we ask for what numbers $a > 0$ we have the following bound for arbitrary initial data $f_N(0)$:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{2d} \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |\omega \cdot (v_2 - v_1)|^2 f_N^{(2)}(t, x_1, v_1, x_1 + \omega, v_2) d\omega dx_1 dv_1 dv_2 dt \leq \\ \leq \frac{C_d}{N^a} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_N^{(1)}(0) dx dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_N^{(1)}(0) dx dv \right)^{\frac{1}{2}} \end{aligned} \quad (294)$$

By Proposition B.1, we have (294) at least for $0 < a \leq 1$.

Proposition B.2. *Let $\varepsilon = 1$. If (294) holds for all $N \in \mathbb{N}$ and arbitrary smooth symmetric probability densities $f_N(0) \in \mathcal{P}(\mathcal{D}_N)$, then $a \leq 1$.*

Proof. Comparing (294) with the equality in (292), it suffices to provide a sequence of initial data $f_N(0)$ (not necessarily smooth) such that the quantities

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot v - |v|^2 T) f_N^{(1)}(T, x, v) dx dv - \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot v f_N^{(1)}(0, x, v) dx dv \quad (295)$$

and

$$\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 f_N^{(1)}(0, x, v) dx dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f_N^{(1)}(0, x, v) dx dv \right)^{\frac{1}{2}} \quad (296)$$

are comparable as $N \rightarrow \infty$ (for some choice of $T = T(N)$). Let N be even and line up all N particles in a row, distributed symmetrically around the origin $0 \in \mathbb{R}^d$. Put a small distance $\delta = \delta(N)$ between neighboring particles¹³; we can send δ to zero as quickly as we like for large N . The

¹³The distance δ does not need to be uniform across the whole array, so we expect that it is possible to avoid multiple collision events globally in time by a measure zero argument. However, we do require that the distribution of particles is symmetric with respect to the

i th particle, with position x_i at $t = 0$, has initial velocity $v_i = -\frac{x_i}{|x_i|}$, so all particles are initially headed towards the origin. By obvious convention we may say that a particle is to the “left” or the “right” of the origin (or another particle), and its velocities may be towards the “left” or to the “right.”

Any collision simply interchanges the velocities of two particles, so the number of particles headed to the left is constant in time, as is the number of particles headed to the right. Moreover, the rightmost left-moving particle never collides after time $\mathcal{O}(N\delta)$.¹⁴ Letting $\delta = o(\frac{1}{N})$, and picking suitable $T \gg N\delta$, we find that the term $\int |v|^2 T f_N^{(1)}(T) dx dv$ in (295) is $o(1)$. After time T , we find that all the left-moving particles are to the left of all the right-moving particles. Since the distribution of particles is initially symmetric about the origin, the same is true at all time. Hence, after time T , all left-moving particles are to the left of the origin, and all right-moving particles are to the right of the origin. Now we can conclude that the integrals $\int x \cdot v f_N^{(1)}(T) dx dv$ and $\int x \cdot v f_N^{(1)}(0) dx dv$ have *opposite signs*, so they cannot cancel each other. Moreover, for small enough δ , $\left| \int x \cdot v f_N^{(1)}(0) dx dv \right|$ is comparable to N by an obvious computation. Hence, the entire expression (295) is comparable to N . The expression (296) is also comparable to N , for small enough δ , by another obvious computation. Hence, (295) and (296) are comparable to each other. \square \square

Remark. If the reader is uncomfortable with the $N\delta$ estimate on the total time of interaction, it is acceptable to set $T = \delta = 0$ at fixed N by a limiting argument. The dynamics is technically not well-defined at the limit *however* any possible combinatorial sequence of binary collisions will result in the same final (scattered) state due to the proof of Proposition B.2. The quantities (295) and (296), correctly interpreted, are both comparable to N .

Modifying slightly the proof of Proposition B.2, it is possible to show that, in the Boltzmann-Grad limit $N\varepsilon^{d-1} = \ell^{-1}$, there exists data $f_N(0)$ such that:

- (i) $f_N^{(1)}(0)$ has uniformly bounded second moments in x and v , and
 - (ii) for each fixed $s \in \mathbb{N}$, the space-time L_{t, Z_s}^1 norm of $C_{s, s+1} f_N^{(s+1)}$ is $\gtrsim \varepsilon^{-1}$.
- In order to produce such an initial data, we line up the particles along $\mathcal{O}\left(N^{\frac{d-2}{d-1}}\right)$ parallel lines all constrained to a $(d-1)$ -dimensional hyperplane in \mathbb{R}^d , instead of a *single* line; of course, each line should contain $\mathcal{O}\left(N^{\frac{1}{d-1}}\right)$ particles. (This example coincides with the proof of Proposition B.2 when $d = 2$.) The velocities are chosen in obvious analogy with the proof of Proposition B.2, so that each line only interacts with itself. It is important to realize that, in this example, the parameter ℓ is not really the mean

origin. In this case we denote by δ the *maximum* distance between neighboring particles at the initial time.

¹⁴To see this, note that the rightmost left-moving particle and the leftmost right-moving particle necessarily collide after a time $\lesssim N\delta$.

free path in any meaningful sense; on the other hand, it is extremely hard to *rule out* the possibility that some “nice” initial condition might lead to a similar physical outcome.¹⁵ The reason for the possible blow-up of the collision operator is that the boundedness of second moments does not prohibit extreme spatial concentration. The possibility of spatial concentration is a well-known issue in the well-posedness of Boltzmann’s equation. [37] The canonical solution to this problem is to use the entropy dissipation and study renormalized solutions [13], but as of this writing, there is no known quantity for deterministic Hamiltonian particle systems which can take the place of entropy dissipation as it appears in Boltzmann’s equation, nor do we have a suitable particle interpretation of renormalized solutions.

B.2. Speculation. We begin by stating the obvious: Proposition B.1, as it stands, is perfectly useless. Since the bound is only in L^1 and *also* non-uniform in N (in the Boltzmann-Grad scaling), this bound cannot possibly give us any information about recollisions or even weak compactness along the interacting set $\{|x_1 - x_2| = \varepsilon\}$, let alone strong chaos.¹⁶ Nevertheless, there are at least two possible directions in which we might hope to improve Proposition B.1.

The first direction would be to revisit the proof of Proposition B.1 at the level of N particles. The proof is based on the monotonicity of the quantity

$$(X_N(t) - tV_N(t)) \cdot V_N(t)$$

along N -particle characteristics. This type of monotonicity is classical for deterministic Hamiltonian particle systems with repulsive interactions. In the context of hard sphere gases, this monotonicity principle has been observed by Illner [20], and was used implicitly by Illner and Pulvirenti for the perturbation of vacuum [22]; similar ideas have also been used by many other authors, among whom we only mention Vaserstein [36] (whose work precedes Illner’s). It is sensible to ask whether other monotonic or convex quantities at the N particle level could be used to obtain better estimates, possibly involving some type of dissipation. Indeed it seems very likely that there is some interaction between *dissipation* and *dispersion* at the particle level [2, 13]; this should be reflected in the uniform bounds we seek to prove.

The second possibility is to recognize that (292) is an *equality* and therefore cannot be improved. Moreover, by a simple computation, it is possible to show that any smooth solution $f(t)$ of Boltzmann’s equation with suitable

¹⁵Then again, it is easy enough to exclude concentration on a *hyperplane* by assuming that the initial data satisfies a suitable spherical symmetry assumption. Most likely, the ε^{-1} bound can still be saturated using spherically symmetric data, but we do not provide an explicit example of this.

¹⁶For one thing, strong chaos is about the behavior the marginals on sets of small measure, not *zero* measure, at a fixed time; whereas, Proposition B.1 is about the behavior of a function on a zero measure set for almost every time.

decay at infinity satisfies the following identity:

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot v - |v|^2 t) f(t, x, v) dx dv = 0 \quad (297)$$

Combining (297) with (292), we conclude that as long as the solution $f(t)$ of Boltzmann's equation remains classical, we have the following bound in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = \ell^{-1}$:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |\omega \cdot (v_2 - v_1)|^2 f_N^{(2)}(t, x_1, v_1, x_1 + \varepsilon\omega, v_2) d\omega dx_1 dv_1 dv_2 dt \\ & \leq C_d \ell \sup_{0 \leq t \leq T} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot v - |v|^2 t) \frac{f_N^{(1)}(t, x, v) - f(t, x, v)}{\varepsilon} dx dv \right| \end{aligned} \quad (298)$$

It is not hard to show that the quantity under the supremum in (298) is uniformly bounded in ε at $t = 0$, for the data constructed in Section 11, as long as $f(0) = f_0$ is smooth and rapidly decaying in all variables.¹⁷ Formally, we also expect that if the solution of Boltzmann's equation remains smooth, then $f_N^{(1)}(t, x, v) = f(t, x, v) + \mathcal{O}(\varepsilon)$ uniformly pointwise, because typical BBGKY pseudo-trajectories are perturbed from the corresponding pseudo-trajectories for the Boltzmann hierarchy by $\mathcal{O}(\varepsilon)$ displacements in space.¹⁸ On the other hand, if the right hand side of (298) remains bounded, then at least formally we extract a new collision bound for the Boltzmann equation. Hence, at least heuristically we have the following logical circle: if $f_N^{(1)}(t)$ remains close to $f(t)$ (at least at the level of conserved quantities) then we can control the interaction uniformly by (298); but, we need to control the interaction *in an even stronger sense* before we can even hope that $f_N^{(1)}(t)$ is close to $f(t)$. In addition to all of this, we must understand how strong chaos fits into the picture. If the logical circle can be closed, we simultaneously obtain a classical solution of Boltzmann's equation (possibly with finite lifetime) *and* validate the model up to the first singularity time (including strong chaos). Note that the validity problem is only slightly simpler if we take for granted the existence of a smooth solution $f(t)$ of Boltzmann's equation.

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¹⁷The bound (298) is in some sense sharp for such initial data; i.e., for any given $T > 0$, the two sides remain bounded or blow up together as $N \rightarrow \infty$, and each side blows up at the same rate in N if there is blow-up at all.

¹⁸Realistically, we probably must lose a logarithm (or worse) in the pointwise estimate, which ruins everything. However, it is reasonable to expect that a special observable like $x \cdot v$ could exhibit a cancellation which recovers the optimal $\mathcal{O}(\varepsilon)$ scaling.

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